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Notation and Conventions

Throughout this work, we adopt the following mathematical conventions:

Scalars: Lowercase italic letters (e.g., t, r, T)

Vectors: Lowercase bold letters (e.g., x, μ)

Matrices: Uppercase bold letters (e.g., Σ, A)

Random variables: Uppercase italic letters (e.g., X, W_t)

Operators: - E[·] = Expectation operator - Var[·] = Variance operator - Cov[·,·] = Covariance operator - ∂/∂x = Partial derivative with respect to x - d/dx = Total derivative with respect to x

Time conventions: - t = Time (years unless otherwise specified) - Δt = Time step - $\mathscr{T}$ = Terminal time (time horizon)
- T = temperature anomaly (°C above pre-industrial)

Financial variables: - r = Discount rate or risk-free rate - V = Asset value - CF = Cash flow - Q = Output or production - K = Capital stock

Climate variables: - T = Temperature anomaly (°C above pre-industrial) - F = Radiative forcing (W/m^2) - M = Carbon mass or concentration - E = Emissions (GtCO_2 or GtC)

Chapter 1: Climate Physics for Financial Modeling

1.1 Introduction

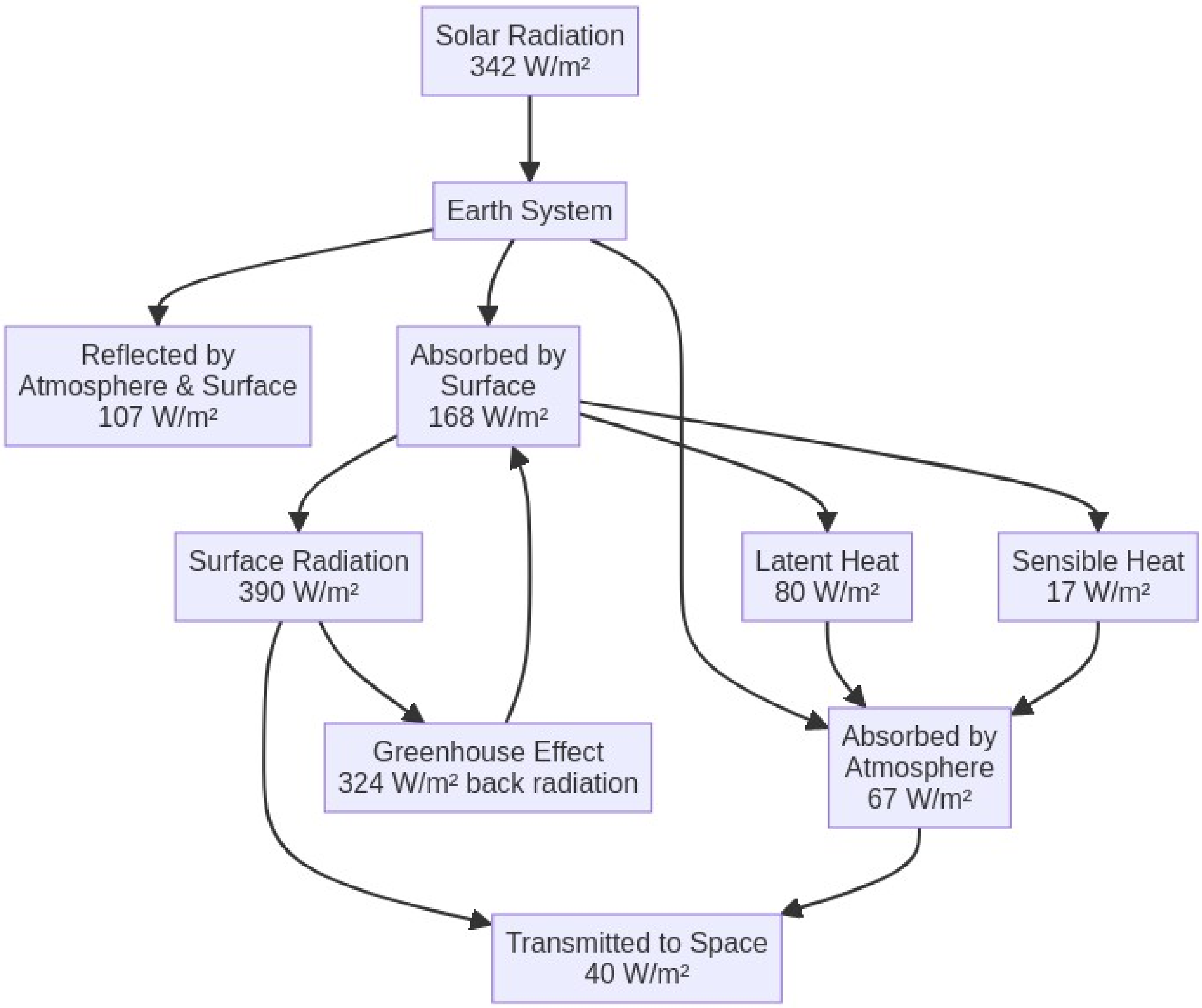

Figure 1.1: Earth's Energy Balance

Figure 1.1: Earth's Energy Balance

The quantification of climate-related financial risk requires a foundational understanding of the physical processes governing Earth's climate system. This chapter establishes the mathematical framework linking greenhouse gas emissions to temperature change, which forms the basis for all subsequent financial modeling. We focus on the energy balance approach, which provides tractable yet scientifically grounded models suitable for integration with economic and financial frameworks [1, 2].

The fundamental insight is that climate change results from an imbalance in Earth's energy budget. Greenhouse gases (GHGs) alter the radiative properties of the atmosphere, trapping outgoing longwave radiation and causing warming. This process can be quantified through the concept of **radiative forcing**, measured in watts per square meter (W/m²) [3].

1.2 Radiative Forcing and the Greenhouse Effect

Definition 1.1 (Radiative Forcing): Radiative forcing is the change in net irradiance at the tropopause after allowing for stratospheric temperatures to readjust to radiative equilibrium, but with surface and tropospheric temperatures held fixed at unperturbed values [3].

The radiative forcing from CO_2 is given by the logarithmic relationship:

$$F_{CO_2} = F_{2x} \cdot \frac{\ln(C/C_0)}{\ln(2)} \text{(Eq. 1.1)}$$

Note: CO_2 forcing parameterization ($F_{2x} \approx 3.71$ W·m^{-2}) follows Myhre et al. (1998); see IPCC AR6 WGI Chapter 7 for context.

where: - F_{CO_2} = radiative forcing from CO_2 (W/m²) - F_{2x} = forcing from doubling CO_2 = 3.71 ± 0.15 W/m² [3, 4] - C = current atmospheric CO_2 concentration (ppm) - C_0 = reference concentration (typically 280 ppm, pre-industrial)

The logarithmic form reflects the saturation of absorption bands: each additional unit of CO_2 has a diminishing marginal effect on forcing [5].

Table 1.1: Radiative Forcing by Greenhouse Gas

Gas	Pre-industrial (ppm)	Current (2023)	Forcing (W/m²)	Lifetime (years)
CO_2	280	420	2.16	300-1000*
CH_4	0.722	1.92	0.54	12.4

N_2O	0.270	0.336	0.21	121
CFC-12	0	0.000503	0.17	100

*CO_2 has no single lifetime; different removal processes operate on different timescales [6].

Source: IPCC AR6 Working Group I [3].

1.2.1 Multi-Gas Forcing

For other greenhouse gases, the forcing relationships differ due to their distinct radiative properties:

Methane (CH_4):

$$F_{CH_4} = 0.036\left(\sqrt{C_{CH_4}} - \sqrt{C_{CH_4,0}}\right) \text{(Eq. 1.1a)}$$

where concentrations are in ppb (parts per billion).

Nitrous Oxide (N_2O):

$$F_{N_2O} = 0.12\left(\sqrt{C_{N_2O}} - \sqrt{C_{N_2O,0}}\right) \text{(Eq. 1.1b)}$$

Total Anthropogenic Forcing:

$$F_{total} = F_{CO_2} + F_{CH_4} + F_{N_2O} + F_{halocarbons} + F_{aerosols} \text{(Eq. 1.1c)}$$

Note that aerosol forcing is negative (cooling effect), partially offsetting GHG warming.

1.3 The Forcing-Feedback Equilibrium Model

The relationship between radiative forcing and equilibrium temperature change is governed by the climate feedback parameter.

Theorem 1.1 (Forcing-Feedback Equilibrium):

At equilibrium, the change in global mean surface temperature ΔT is related to radiative forcing F by:

$$\Delta T_{eq} = \frac{F}{\lambda} \text{(Eq. 1.2)}$$

where λ is the climate feedback parameter (W/m²/K).

Proof:

- At equilibrium, the change in net radiation at the top of the atmosphere must be zero.
- The radiative forcing F represents the initial perturbation to the energy balance.
- As the surface warms by ΔT, the system responds through various feedbacks (Planck response, water vapor, lapse rate, albedo, clouds). The total feedback can be linearized as:

$$\Delta R_{feedback} = -\lambda \cdot \Delta T \text{(Eq. 1.3)}$$

where the negative sign indicates that positive ΔT leads to increased outgoing radiation (negative feedback from Planck response dominates).

(d) At equilibrium: $F + \Delta R_{feedback} = 0$

(e) Substituting Eq. 1.3: $F - \lambda \cdot \Delta T = 0$

(f) Solving for ΔT yields Eq. 1.2. ∎

The climate feedback parameter λ can be decomposed into individual feedback components:

$$\lambda = \lambda_{Planck} + \lambda_{WV} + \lambda_{LR} + \lambda_{albedo} + \lambda_{cloud} \text{(Eq. 1.4)}$$

Table 1.2: Climate Feedback Components

Feedback	Symbol	Value (W/m²/K)	Sign	Physical Mechanism
Planck	λ_P	-3.2	Negative	Stefan-Boltzmann radiation
Water Vapor	λ_{WV}	+1.8	Positive	Increased atmospheric H_2O
Lapse Rate	λ_{LR}	-0.5	Negative	Tropospheric warming profile
Albedo	λ_{alb}	+0.4	Positive	Ice/snow melt reduces reflectivity
Cloud	λ_{cloud}	+0.4	Positive	Net cloud feedback (uncertain)
Total	λ	-1.1	**Negative**	**Net stabilizing**

Source: IPCC AR6 [3, Chapter 7].

Important Note on Sign Convention: The net feedback parameter λ = -1.1 $W/m^2/K$ is negative, which represents a net stabilizing (negative) feedback. The negative sign arises because the Planck response (λ_P = -3.2) dominates the positive feedbacks. When $\lambda < 0$, we write Eq. 1.2 as:

$$\Delta T_{eq} = \frac{F}{¿\lambda \vee ¿ = \frac{F}{1.1}(\text{with } \lambda = \text{-}1.1)¿}$$

This ensures $\Delta T > 0$ for $F > 0$, as physically required.

1.4 Climate Sensitivity

Definition 1.2 (Equilibrium Climate Sensitivity): The equilibrium climate sensitivity (ECS) is the equilibrium change in global mean surface temperature following a doubling of atmospheric CO_2 concentration [3].

From Eq. 1.1 and 1.2, with $C = 2C_0$:

$$ECS = \frac{F_{2x}}{¿\lambda \vee ¿(\text{Eq. 1.5})¿}$$

IPCC AR6 Assessment [3]: - Best estimate: ECS = 3.0°C - Likely range: 2.5°C to 4.0°C (66% probability) - Very likely range: 2.0°C to 5.0°C (90% probability)

This represents a significant narrowing of uncertainty compared to previous assessments, achieved through multiple lines of evidence including paleoclimate records, historical observations, and process understanding [7].

Corollary 1.1: The no-feedback climate response (if $\lambda = \lambda_{Planck}$ only) would be:

$$\Delta T_{no-feedback} = \frac{F_{2x}}{¿\lambda_{Planck} \vee ¿ = \frac{3.71}{3.2} \approx 1.16\,°C\,¿}$$

The ratio ECS / $\Delta T_{no\text{-}feedback}$ ≈ 2.6 quantifies the amplification from positive feedbacks.

1.5 Transient Climate Response

Equilibrium climate sensitivity describes the long-term steady state, but financial risk assessment requires understanding the transient response on decision-relevant timescales (decades).

Definition 1.3 (Transient Climate Response): The transient climate response (TCR) is the change in global mean surface temperature at the time of CO_2 doubling in a scenario where CO_2 increases at 1% per year [3].

TCR is always less than ECS because: 1. The deep ocean has not equilibrated 2. Heat is still being absorbed by the climate system

Relationship: Empirically, TCR ≈ 0.6 × ECS [8].

IPCC AR6 Assessment [3]: - Best estimate: TCR = 1.8°C - Likely range: 1.4°C to 2.2°C

For financial modeling, TCR is more relevant than ECS for projections to 2050-2100.

1.5.1 Two-Layer Energy Balance Model

To capture transient dynamics, we extend the simple equilibrium model to a two-layer system representing the upper ocean/atmosphere and deep ocean [2]:

Upper layer (fast response):

$$C_1 \frac{dT_1}{dt} = F - \lambda T_1 - \gamma (T_1 - T_2) \text{(Eq. 1.6)}$$

Deep layer (slow response):

$$C_2 \frac{dT_2}{dt} = \gamma (T_1 - T_2) \text{(Eq. 1.7)}$$

where: - T_1 = upper layer temperature anomaly (°C) - T_2 = deep ocean temperature anomaly (°C) - C_1 = heat capacity of upper layer ≈ 8 $W{\cdot}yr{\cdot}m^{-2}{\cdot}K^{-1}$ - C_2 = heat capacity of deep ocean ≈ 100 $W{\cdot}yr{\cdot}m^{-2}{\cdot}K^{-1}$ - γ = ocean heat uptake coefficient ≈ 0.7 $W{\cdot}m^{-2}{\cdot}K^{-1}$

This system exhibits two timescales: - Fast timescale: $\tau_1 \approx C_1/\lambda \approx$ 7 years - Slow timescale: $\tau_2 \approx C_2/\gamma \approx$ 140 years

1.6 Carbon Cycle Dynamics

The relationship between emissions and atmospheric concentration requires modeling the carbon cycle. We present a simplified three-box model suitable for financial applications [9].

Model Structure:

The carbon cycle is represented by three reservoirs: - Atmosphere (M_{atm}) - Upper ocean and terrestrial biosphere (M_{upper}) - Deep ocean (M_{deep})

Governing Equations:

$$\frac{dM_{atm}}{dt} = E(t) - k_1(M_{atm} - M_{atm,eq}) - k_2(M_{atm} - M_{upper}) \text{(Eq. 1.8)}$$

$$\frac{dM_{upper}}{dt} = k_2(M_{atm} - M_{upper}) - k_3(M_{upper} - M_{deep}) \text{(Eq. 1.9)}$$

$$\frac{dM_{deep}}{dt} = k_3(M_{upper} - M_{deep}) \text{(Eq. 1.10)}$$

Calibration note: For policy analysis, simple box models should be calibrated against impulse-response functions (IRFs) such as Joos et al. (2013). The historical airborne fraction (~0.44) is not constant as sinks evolve (Global Carbon Budget).

where: - E(t) = anthropogenic emissions (GtC/year) - k_1 = 0.02 $year^{-1}$ (land uptake rate) - k_2 = 0.05 $year^{-1}$ (atmosphere-upper ocean exchange) - k_3 = 0.003 $year^{-1}$ (upper-deep ocean exchange) - $M_{atm,eq}$ = equilibrium atmospheric carbon (588 GtC for pre-industrial)

Airborne Fraction:

The fraction of emitted CO_2 remaining in the atmosphere is:

$$AF(t) = \frac{M_{atm}(t) - M_{atm}(0)}{\int_0^t E(s)\,ds} \text{(Eq. 1.11)}$$

Historically, AF ≈ 0.44 (44% remains airborne, 56% absorbed by land and ocean sinks) [10].

Important Note: The airborne fraction may increase over time as sinks saturate, creating a positive feedback [11]. This is not captured in the simple linear model above but is included in comprehensive Earth System Models.

1.7 Worked Examples

Example 1.1: Calculating Radiative Forcing

Problem: Calculate the current radiative forcing from CO_2 given: - Pre-industrial concentration: C_0 = 280 ppm - Current concentration: C = 420 ppm - F_{2x} = 3.71 W/m²

Solution:

Using Eq. 1.1:

$$F_{CO_2} = 3.71 \times \frac{\ln(420/280)}{\ln(2)} = 3.71 \times \frac{0.4055}{0.6931} = 3.71 \times 0.585 = 2.17 \text{ W/m}^2$$

This matches the IPCC AR6 estimate of 2.16 W/m² [3]. ∎

Example 1.2: Equilibrium Temperature from Forcing

Problem: Given the forcing calculated above and λ = -1.1 W/m²/K, calculate the equilibrium temperature change.

Solution:

Using Eq. 1.2:

$$\Delta T_{eq} = \frac{F}{\text{¿}\lambda \vee \text{¿} = \frac{2.17}{1.1} = 1.97\,°C\,\text{¿}}$$

This represents the committed warming from current CO_2 levels alone (excluding other GHGs and assuming equilibrium is reached). ∎

Example 1.3: Projecting Future Concentrations

Problem: If emissions remain constant at E = 10 GtC/year and the airborne fraction is 0.44, how much will atmospheric CO_2 increase in 30 years?

Solution:

Atmospheric increase = AF × Total emissions

$$\Delta M_{atm} = 0.44 \times (10 \text{ GtC/year} \times 30 \text{ years}) = 132 \text{ GtC}$$

Converting to ppm (1 ppm ≈ 2.13 GtC):

$$\Delta C = \frac{132}{2.13} = 62 \text{ ppm}$$

Future concentration: C(2055) = 420 + 62 = 482 ppm. ∎

Example 1.4: Multi-Gas Forcing Calculation

Problem: Calculate the total radiative forcing in 2023 from CO_2, CH_4, and N_2O given: - CO_2: 280 ppm → 420 ppm - CH_4: 722 ppb → 1920 ppb - N_2O: 270 ppb → 336 ppb

Solution:

CO_2 forcing (from Example 1.1):

$$F_{CO_2} = 2.17 \text{ W/m}^2$$

CH_4 forcing using Eq. 1.1a:

$$F_{CH_4} = 0.036(\sqrt{1920} - \sqrt{722}) = 0.036(43.82 - 26.87) = 0.036 \times 16.95 = 0.61 \text{ W/m}^2$$

N_2O forcing using Eq. 1.1b:

$$F_{N_2O} = 0.12(\sqrt{336} - \sqrt{270}) = 0.12(18.33 - 16.43) = 0.12 \times 1.90 = 0.23 \text{ W/m}^2$$

Total forcing:

$$F_{total} = 2.17 + 0.61 + 0.23 = 3.01 \text{ W/m}^2$$

This is consistent with IPCC AR6 estimates of ~3.0 W/m² for well-mixed greenhouse gases. ∎

Example 1.5: RCP8.5 Forcing Trajectory

Problem: Calculate the radiative forcing in 2100 under RCP8.5, which projects CO_2 concentration of 936 ppm.

Solution:

Using Eq. 1.1 with C_0 = 280 ppm and C = 936 ppm:

$$F_{CO_2}(2100) = 3.71 \times \frac{\ln(936/280)}{\ln(2)} = 3.71 \times \frac{1.205}{0.693} = 3.71 \times 1.739 = 6.45 \text{ W/m}^2$$

This is approximately 1.74 CO_2 doublings (since 936/280 ≈ 3.34 = 2^1.74).

Including other GHGs and aerosols, RCP8.5 reaches total forcing of ~8.5 W/m² by 2100 (hence the name). ∎

Example 1.6: Regional Temperature Scaling

Problem: If global mean temperature increases by 2.0°C, estimate the temperature increase over land and ocean separately, given that land warms ~1.6 times faster than the global mean and ocean warms ~0.7 times the global mean.

Solution:

Land warming:

$$\Delta T_{land} = 1.6 \times \Delta T_{global} = 1.6 \times 2.0 = 3.2°C$$

Ocean warming:

$$\Delta T_{ocean} = 0.7 \times \Delta T_{global} = 0.7 \times 2.0 = 1.4°C$$

Verification (area-weighted average): With land fraction f_{land} ≈ 0.29 and ocean fraction f_{ocean} ≈ 0.71:

$$\Delta T_{global} = f_{land} \cdot \Delta T_{land} + f_{ocean} \cdot \Delta T_{ocean}$$

$$¿ 0.29 \times 3.2 + 0.71 \times 1.4 = 0.93 + 0.99 = 1.92\,°C \approx 2.0\,°C$$

✓

This regional heterogeneity is critical for financial risk assessment, as most economic activity occurs on land. ∎

Example 1.7: Ocean Heat Uptake Efficiency

Problem: Using the two-layer model (Eqs. 1.6-1.7), calculate the ocean heat uptake rate when T_1 = 1.2°C and T_2 = 0.3°C, with γ = 0.7 W·m^{-2}·K^{-1}.

Solution:

The ocean heat uptake is given by the heat flux from upper to deep ocean:

$$Q_{ocean} = \gamma (T_1 - T_2) = 0.7 \times (1.2 - 0.3) = 0.7 \times 0.9 = 0.63\ \text{W/m}^2$$

This represents the rate at which heat is being absorbed by the deep ocean, delaying surface warming. Over the entire Earth surface (5.1 × 10^{14} m^2):

$$\text{Total heat uptake} = 0.63 \times 5.1 \times 10^{14} = 3.2 \times 10^{14}\ \text{W} = 320\ \text{TW}$$

This is equivalent to the energy of ~200,000 nuclear power plants continuously operating. ∎

Example 1.8: Carbon Budget for 1.5°C Target

Problem: Calculate the remaining carbon budget to limit warming to 1.5°C above pre-industrial, given: - Current warming: 1.1°C - TCR = 1.8°C - TCRE (Transient Climate Response to cumulative CO_2 Emissions) = 0.45°C per 1000 $GtCO_2$

Solution:

Remaining warming budget:

$$\Delta T_{remaining} = 1.5 - 1.1 = 0.4\,°C$$

Remaining carbon budget:

$$B_{remaining} = \frac{\Delta T_{remaining}}{TCRE} = \frac{0.4}{0.45/1000} = \frac{0.4 \times 1000}{0.45} = 889\ \text{GtCO}_2$$

At current emissions rate (40 $GtCO_2$/year):

$$\text{Years remaining} = \frac{889}{40} = 22.2\ \text{years}$$

This suggests the 1.5°C budget would be exhausted around 2047 at current emission rates, highlighting the urgency of mitigation. This calculation is fundamental for financial scenario analysis and stranded asset risk. ∎

1.8 Supplementary Problems

Basic Problems (1-5)

- **Derive the relationship between the feedback parameter λ and climate sensitivity S = 1/λ.** Show that small changes in λ lead to large changes in S when λ is small. Specifically, prove that:

$$\frac{dS}{d\lambda} = \frac{-1}{\lambda^2}$$

- and evaluate this derivative at λ = 1.0 and λ = 0.5 W/m²/K.

- **Calculate the forcing from a 50% increase in methane concentration** (from 1.92 to 2.88 ppm) using the formula: $F_{CH_4} = 0.036 \times (\sqrt{C} - \sqrt{C_0})$ W/m². Compare this to the forcing from a 50% increase in CO_2 (from 420 to 630 ppm).

- **Prove that the three-box carbon cycle model (Eqs. 1.8-1.10) conserves total carbon mass.** Show that:

$$\frac{d}{dt}(M_{atm}+M_{upper}+M_{deep})=E(t)$$

- **Estimate the time constant for atmospheric CO_2 to equilibrate with the upper ocean** ($\tau = 1/k_2$). Given k_2 = 0.05 $year^{-1}$, calculate τ and interpret the result.
- **Using the transient climate response (TCR = 1.8°C), estimate the temperature change in 2050** under RCP4.5 (which reaches ~500 ppm CO_2-equivalent by 2050). Assume linear relationship between forcing and TCR.

Intermediate Problems (6-10)

(f) **Decompose the climate feedback parameter. Given individual feedbacks from Table 1.2, verify that λ_{total} = -1.1 W/m²/K. Then calculate the "feedback factor" $f = 1/(1 - \Sigma g_i)$ where $g_i = -\lambda_i/\lambda_{Planck}$ for each feedback component.**

(g) **Solve the two-layer energy balance model analytically** for the case of constant forcing F applied at t = 0. Find $T_1(t)$ and $T_2(t)$ assuming initial conditions $T_1(0) = T_2(0) = 0$.

(h) **Calculate the airborne fraction evolution.** If land and ocean sinks saturate such that k_1 decreases from 0.02 to 0.015 $year^{-1}$ and k_2 decreases from 0.05 to 0.04 $year^{-1}$, how does the steady-state airborne fraction change?

(i) **Estimate the "committed warming"** from current CO_2 levels. If we stopped all emissions today (E = 0), how much additional warming would occur as the system equilibrates? Use ECS = 3.0°C and current forcing F = 2.17 W/m².

(j) **Derive the relationship between ECS and TCR** using the two-layer model. Show that TCR/ECS = $(1 + \kappa)/(1 + \kappa \cdot C_2/C_1)$ where $\kappa = \gamma/\lambda$.

Advanced Problems (11-15)

(k) **Uncertainty propagation in ECS.** Given F_{2x} = 3.71 ± 0.15 W/m² and λ = -1.1 ± 0.3 W/m²/K (both normally distributed), calculate the probability distribution of ECS using:

   (a) First-order error propagation

(b) Monte Carlo simulation (10,000 samples) Compare the results and explain any differences.

(l) **Non-linear carbon cycle feedback.** Modify the three-box model to include temperature-dependent sink rates: $k_1(T) = k_{1,0}(1 - \alpha T)$ where $\alpha = 0.05\ K^{-1}$. Solve numerically for atmospheric CO_2 concentration from 2020-2100 under RCP8.5 emissions and compare to the linear case.

(m) **Regional climate patterns.** Using the land/ocean warming ratio from Example 1.6, derive a simple model for continental interior warming as a function of distance from coast. Assume warming ratio varies as: R(d) = 0.7 + 0.9(1 - e^(-d/L)) where d is distance from coast and L = 1000 km.

(n) **Paleoclimate constraint on ECS.** The Last Glacial Maximum (LGM, 21,000 years ago) had global temperature 5°C colder than pre-industrial and CO_2 concentration of 180 ppm (vs. 280 ppm pre-industrial). Use this to estimate ECS, accounting for ice sheet albedo feedback (additional -3.5 W/m² forcing during LGM).

(o) **Tipping point analysis. Consider a simplified model where the ice-albedo feedback becomes unstable if $T > T_{crit} = 2.5$°C. Model this as a regime change where λ_{albedo} increases from +0.4 to +1.2 W/m²/K when $T > T_{crit}$. Calculate the new equilibrium temperature under F = 4 W/m² forcing and discuss implications for financial risk assessment.**

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Chapter 2: Financial Mathematics for Climate Risk

2.1 Principles of Asset Valuation: The Discounted Cash Flow (DCF) Model

The cornerstone of modern finance is the principle that the value of an asset is the present value of its expected future cash flows. The Discounted Cash Flow (DCF) model is the canonical mathematical formulation of this principle.

Definition 2.1 (Standard DCF Model): The value of an asset (V_0) at time t=0 is the sum of all future expected cash flows (CF_t) from t=1 to T, discounted back to the present at a specified discount rate (r):

$$V_0 = \sum_{t=1}^{T} \frac{E[CF_t]}{¿¿} ¿$$

Where: - $E[CF_t]$ is the expected cash flow in period t - r is the discount rate, reflecting the time value of money and the riskiness of the cash flows - t is the time period

For perpetual cash flows ($T \to \infty$), the formula simplifies to a perpetuity or growing perpetuity model, depending on the assumptions about cash flow growth.

Corollary 2.1 (Perpetuity Formula): For constant perpetual cash flows CF:

$$V_0 = \frac{CF}{r} \text{(Eq. 2.2)}$$

Corollary 2.2 (Growing Perpetuity Formula): For cash flows growing at constant rate $g < r$:

$$V_0 = \frac{CF_1}{r-g} \text{(Eq. 2.3)}$$

2.2 Climate-Adjusted Discounted Cash Flow Model

Theorem 2.1 (Climate-Adjusted DCF)

Statement: The value of an asset subject to climate risk is the present value of its expected future cash flows, adjusted by a climate damage function $D(T_t)$ that quantifies the fractional reduction in cash flows due to the physical impacts of climate change at time t.

$$V_0=\sum_{t=1}^{T}\frac{E[CF_t]\cdot(1-D(T_t))}{¿¿}¿$$

Where: - $D(T_t)$ is the climate damage function, which maps the projected temperature anomaly (T_t) at time t to a fractional economic loss ($0 \leq D(T_t) \leq 1$) - r_c is the climate-adjusted discount rate, which may include a premium for climate-related risks

Proof:

- Let CF_t be the baseline expected cash flow at time t in a world without climate change.
- Let T_t be the projected global mean temperature anomaly at time t, derived from a physical climate model as described in Chapter 1.
- Let $D(T_t)$ be a continuous, non-decreasing function representing the fractional damage to economic output caused by the temperature anomaly T_t. The existence of such functions is empirically supported [1].
- The climate-impacted cash flow at time t, CF'_t, is the baseline cash flow reduced by the climate damage:

$$CF'_t=CF_t-CF_t\cdot D(T_t)=CF_t\cdot(1-D(T_t))$$

- The expected value of the climate-impacted cash flow is:

$$E[CF'_t]=E[CF_t\cdot(1-D(T_t))]$$

Assuming the damage function is deterministic for a given temperature path (or taking expectations over both CF and T):

$$E[CF'_t]=E[CF_t]\cdot(1-E[D(T_t)])$$

For notational simplicity, we write $D(T_t)$ to represent $E[D(T_t)]$.

- The value of the asset is the sum of the present values of these climate-impacted cash flows, discounted at a climate-adjusted rate r_c:

$$V_0=\sum_{t=1}^{T}\frac{E[CF'_t]}{¿¿}¿$$

This completes the proof. ∎

Corollary 2.3 (Climate Impact on Asset Value): The fractional reduction in asset value due to climate change is:

$$\frac{V_0^{baseline} - V_0^{climate}}{V_0^{baseline}} = 1 - \sum_{t=1}^{T} \frac{CF_t(1 - D(T_t))}{¿¿}¿¿$$

2.3 Financial Risk Metrics

To quantify the potential losses from climate change, we employ standard financial risk metrics, adapted for this context.

2.3.1 Value-at-Risk (VaR)

Definition 2.2 (Value-at-Risk): Value-at-Risk (VaR) is the maximum potential loss on a portfolio over a given time horizon within a given confidence level (c). Formally, for a loss L, VaR_c is the value such that:

$$P(L > VaR_c) = 1 - c \text{ (Eq. 2.6)}$$

Equivalently:

$$P(L \leq VaR_c) = c$$

If the portfolio losses are normally distributed with mean μ and standard deviation σ, the VaR can be calculated directly:

$$VaR_c = \mu + \sigma \cdot Z_c \text{(Eq. 2.7)}$$

where Z_c is the c-quantile of the standard normal distribution.

Common confidence levels: - 95%: $Z_{0.95}$ = 1.645 - 99%: $Z_{0.99}$ = 2.326 - 99.9%: $Z_{0.999}$ = 3.090

2.3.2 Expected Shortfall (ES)

Definition 2.3 (Expected Shortfall): Expected Shortfall, also known as Conditional VaR (CVaR), measures the expected loss given that the loss exceeds the VaR. It provides a measure of the magnitude of tail losses.

$$ES_c = E[L \mid L > VaR_c] \text{(Eq. 2.8)}$$

For a normally distributed loss, the ES is given by:

$$ES_c = \mu + \sigma \cdot \frac{\phi(Z_c)}{1-c} \text{(Eq. 2.9)}$$

where φ(z) is the probability density function (PDF) of the standard normal distribution:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

2.4 The Term Structure of Climate Risk

The discount rate used in valuation should, in theory, account for the systematic risks associated with climate change. This can be modeled by incorporating a time-varying climate risk premium into the discount rate.

The climate-adjusted discount rate, $r_c(t)$, can be modeled as:

$$r_c(t) = r_f + \beta \cdot MRP + rp_{climate}(t) \text{(Eq. 2.10)}$$

Where: - r_f is the risk-free rate - β is the asset beta (systematic market risk) - MRP is the market risk premium - $rp_{climate}(t)$ is the climate risk premium at time t

Theorem 2.2 (Climate Risk Premium): The climate risk premium can be derived from the covariance of asset returns with climate damages:

$$rp_{climate} = \frac{Cov(R_i, D)}{Var(R_m)} \cdot MRP \text{(Eq. 2.11)}$$

where R_i is the asset return, D is climate damage, and R_m is market return.

Proof: (Sketch)

- From CAPM, the required return on asset i is: $r_i = r_f + \beta_i \cdot MRP$
- Climate risk introduces an additional systematic factor. Using multi-factor model:

$$r_i = r_f + \beta_{market} \cdot MRP + \beta_{climate} \cdot CRP$$

- The climate beta is:

$$\beta_{climate} = \frac{Cov(R_i, D)}{Var(D)}$$

- The climate risk premium (CRP) is proportional to market risk premium by the ratio of climate risk to market risk:

$$CRP = MRP \cdot \frac{Var(D)}{Var(R_m)}$$

- Combining: $rp_{climate} = \beta_{climate} \cdot CRP$ yields Eq. 2.11. ∎

2.5 Worked Examples

Example 2.1: Climate-Adjusted DCF Calculation

Problem: An asset is expected to generate a perpetual cash flow of \$100 per year. The discount rate is 8%. A climate model projects that damages will be 5% of cash flows in perpetuity. Calculate the asset value with and without climate impacts.

Solution:

Without Climate Impacts: Using the perpetuity formula (Eq. 2.2):

$$V_0 = \frac{CF}{r} = \frac{100}{0.08} = \$1{,}250$$

With Climate Impacts: The climate-adjusted cash flow is:

$$CF' = CF \cdot (1 - D) = 100 \cdot (1 - 0.05) = \$95$$

$$V_0^{climate} = \frac{95}{0.08} = \$1{,}187.50$$

Climate Impact:

$$\text{Value loss} = 1{,}250 - 1{,}187.50 = \$62.50$$

$$\text{Percentage loss} = \frac{62.50}{1{,}250} = 5\%$$

The climate impact causes a valuation loss of \$62.50, or 5%. ∎

Example 2.2: Calculating Climate VaR

Problem: A portfolio's value is projected to be impacted by climate change. A Monte Carlo simulation (see Chapter 5) of 10,000 scenarios yields a distribution of climate-related losses with a mean (μ) of \$50 million and a standard deviation (σ) of \$150 million. Assuming a normal distribution, calculate the 99% VaR.

Solution:

- The confidence level c = 0.99
- The Z-score corresponding to 99% confidence is $Z_{0.99}$ = 2.326
- Calculate the VaR using Eq. 2.7:

$$VaR_{99\%} = \mu + \sigma \cdot Z_c = 50 + 150 \cdot 2.326 = 50 + 348.9 = \$398.9 \text{ million}$$

This means there is a 1% chance that the portfolio's climate-related losses will exceed \$398.9 million. ∎

Example 2.3: Time-Varying Climate Damages

Problem: An asset generates cash flows of \$1,000 per year for 30 years. Climate damages are projected to increase linearly from 0% in year 1 to 15% in year 30. The discount rate is 6%. Calculate the present value with climate impacts.

Solution:

The damage function is:

$$D(t) = 0.15 \cdot \frac{t}{30} = 0.005t$$

The climate-adjusted cash flow in year t is:

$$CF'_t = 1{,}000 \cdot (1 - 0.005t)$$

The present value is:

$$V_0 = \sum_{t=1}^{30} \frac{1{,}000\left(1 - 0.005\,t\right)}{\text{¿¿}} \text{¿}$$

Separating terms:

$$V_0 = 1{,}000 \sum_{t=1}^{30} \frac{1}{\text{¿¿}} \text{¿}$$

The first sum is a standard annuity:

$$\sum_{t=1}^{30} \frac{1}{\text{¿¿}} \text{¿}$$

The second sum requires the formula: $\sum_{t=1}^{n} \frac{t}{\text{¿¿}} \text{¿}$

$$\sum_{t=1}^{30} \frac{t}{\text{¿¿}} \text{¿}$$

Therefore:

$$V_0 = 1{,}000 \cdot 13.765 - 5 \cdot 142.35 = 13{,}765 - 711.75 = \$13{,}053.25$$

Baseline value (no climate damage):

$$V_0^{baseline} = 1{,}000 \cdot 13.765 = \$13{,}765$$

Climate impact:

$$\text{Value loss} = 13{,}765 - 13{,}053.25 = \$711.75$$

$$\text{Percentage loss} = \frac{711.75}{13{,}765} = 5.17\,\%$$

Despite damages reaching 15% by year 30, the present value loss is only 5.17% due to discounting. ∎

Example 2.4: Expected Shortfall Calculation

Problem: For the portfolio in Example 2.2, calculate the 99% Expected Shortfall (ES).

Solution:

Given: - μ = \$50 million - σ = \$150 million - c = 0.99 - $Z_{0.99}$ = 2.326

Using Eq. 2.9:

$$ES_{99\%} = \mu + \sigma \cdot \frac{\phi(Z_c)}{1-c}$$

Calculate φ(2.326):

$$\phi(2.326) = \frac{1}{\sqrt{2\pi}} e^{-¿¿}$$

Therefore:

$$ES_{99\%} = 50 + 150 \cdot \frac{0.0267}{0.01} = 50 + 150 \cdot 2.67 = 50 + 400.5 = \$450.5 \text{ million}$$

Interpretation: Given that losses exceed the 99% VaR (\$398.9M), the expected loss is \$450.5M. The difference (\$51.6M) represents the expected excess loss in the worst 1% of scenarios. ∎

Example 2.5: Growing Perpetuity with Climate Damages

Problem: An asset generates cash flows of \$1,000 next year, growing at 3% per year. The discount rate is 8%. Climate damages are 2% of cash flows in perpetuity. Calculate the climate-adjusted value.

Solution:

Without climate damages: Using Eq. 2.3:

$$V_0^{baseline} = \frac{CF_1}{r-g} = \frac{1{,}000}{0.08-0.03} = \frac{1{,}000}{0.05} = \$20{,}000$$

With climate damages: The climate-adjusted cash flow is:

$$CF'_1 = 1{,}000 \cdot (1 - 0.02) = \$980$$

$$V_0^{climate} = \frac{980}{0.08 - 0.03} = \frac{980}{0.05} = \$19{,}600$$

Climate impact:

$$\text{Value loss} = 20{,}000 - 19{,}600 = \$400$$

$$\text{Percentage loss} = \frac{400}{20{,}000} = 2\%$$

The percentage value loss equals the damage percentage for perpetual constant damages. ∎

Example 2.6: Climate Risk Premium Estimation

Problem: An equity portfolio has the following characteristics: - Market beta (β_{market}) = 1.2 - Correlation with climate damages ($\rho_{R,D}$) = 0.3 - Volatility of returns (σ_R) = 20% - Volatility of climate damages (σ_D) = 15% - Market risk premium (MRP) = 7% - Market volatility (σ_m) = 18%

Calculate the climate risk premium.

Solution:

The covariance of returns with climate damages is:

$$Cov(R, D) = \rho_{R,D} \cdot \sigma_R \cdot \sigma_D = 0.3 \cdot 0.20 \cdot 0.15 = 0.009$$

The variance of market returns is:

$$Var(R_m) = \sigma_m^2 = ¿$$

Using Eq. 2.11:

$$r\,p_{climate} = \frac{Cov(R, D)}{Var(R_m)} \cdot MRP = \frac{0.009}{0.0324} \cdot 0.07 = 0.278 \cdot 0.07 = 0.0194 = 1.94\%$$

The total required return is:

$$r_c = r_f + \beta_{market} \cdot MRP + r\,p_{climate}$$

Assuming $r_f = 3\%$:

$$r_c = 0.03 + 1.2 \cdot 0.07 + 0.0194 = 0.03 + 0.084 + 0.0194 = 13.34\,\%$$

The climate risk premium adds 194 basis points to the required return. ∎

Example 2.7: Multi-Period DCF with Scenario Analysis

Problem: A project generates cash flows of \$500, \$600, \$700, \$800, \$900 over 5 years. Analyze three climate scenarios: - Base case: No damages (probability 40%) - Moderate: 10% damage starting year 3 (probability 45%) - Severe: 25% damage starting year 2 (probability 15%)

Discount rate is 8%. Calculate the expected NPV.

Solution:

Base case NPV:

$$NPV_{base} = \sum_{t=1}^{5} \frac{CF_t}{¿¿} ¿$$

$$¿ 462.96 + 514.40 + 555.87 + 588.07 + 612.52 = \$2{,}733.82$$

Moderate case NPV: Cash flows: \$500, \$600, \$630 (=700×0.9), \$720 (=800×0.9), \$810 (=900×0.9)

$$NPV_{moderate} = 462.96 + 514.40 + 500.28 + 529.26 + 551.27 = \$2{,}558.17$$

Severe case NPV: Cash flows: \$500, \$450 (=600×0.75), \$525 (=700×0.75), \$600 (=800×0.75), \$675 (=900×0.75)

$$NPV_{severe} = 462.96 + 385.80 + 416.90 + 441.05 + 459.39 = \$2{,}166.10$$

Expected NPV:

$$E[NPV] = 0.40 \cdot 2{,}733.82 + 0.45 \cdot 2{,}558.17 + 0.15 \cdot 2{,}166.10$$

$$¿1{,}093.53+1{,}151.18+324.92=\$2{,}569.63$$

Climate impact:

$$\text{Value loss}=2{,}733.82-2{,}569.63=\$164.19$$

$$\text{Percentage loss}=\frac{164.19}{2{,}733.82}=6.0\%$$

The expected climate impact reduces NPV by 6.0%. ∎

Example 2.8: WACC Adjustment for Climate Risk

Problem: A company has: - Cost of equity (r_e) = 12% - Cost of debt (r_d) = 5% - Tax rate (τ) = 25% - Debt-to-equity ratio (D/E) = 0.5

Climate risk analysis suggests adding a 150 bp climate risk premium to the cost of equity. Calculate the baseline and climate-adjusted WACC.

Solution:

The weights are:

$$w_e=\frac{E}{D+E}=\frac{1}{1+0.5}=0.667$$

$$w_d=\frac{D}{D+E}=\frac{0.5}{1.5}=0.333$$

Baseline WACC:

$$WACC=w_e\cdot r_e+w_d\cdot r_d\cdot(1-\tau)$$

$$¿0.667\cdot 0.12+0.333\cdot 0.05\cdot 0.75$$

$$¿0.0800+0.0125=0.0925=9.25\%$$

Climate-adjusted WACC:

$$r_e^{climate}=0.12+0.015=0.135=13.5\%$$

$$WACC^{climate} = 0.667 \cdot 0.135 + 0.333 \cdot 0.05 \cdot 0.75$$

$$¿ 0.0900 + 0.0125 = 0.1025 = 10.25\%$$

Impact on valuation: For a perpetual cash flow of \$100M:

$$V_{baseline} = \frac{100}{0.0925} = \$1{,}081\,M$$

$$V_{climate} = \frac{100}{0.1025} = \$976\,M$$

$$\text{Value loss} = \frac{1{,}081 - 976}{1{,}081} = 9.7\%$$

A 100 bp increase in WACC reduces firm value by 9.7%. ∎

2.6 Supplementary Problems

Basic Problems (1-6)

- An asset is expected to generate \$1,000 in cash flow next year, growing at 2% in perpetuity. The discount rate is 10%. Climate damages are projected to reduce cash flows by 3% permanently. Calculate the percentage reduction in the asset's value due to climate change.

- For a normally distributed loss with mean \$100M and standard deviation \$50M, calculate the 95% VaR and compare it to the 99% VaR.

- Prove that for a given asset, if the climate risk premium ($rp_{climate}$) increases, the asset's value will decrease. Use the perpetuity formula to demonstrate.

- A 10-year bond pays annual coupons of \$50 and has a face value of \$1,000. If climate risk adds 50 bp to the discount rate (from 5% to 5.5%), calculate the change in bond value.

- Calculate the 95% Expected Shortfall for a loss distribution with μ = \$20M and σ = \$30M.

- An asset has cash flows of \$100, \$110, \$121 over 3 years (growing at 10%). Climate damages are 5% in all years. Discount rate is 8%. Calculate the climate-adjusted NPV.

Intermediate Problems (7-12)

7. Derive the formula for the climate-adjusted growing perpetuity: $V_0 = \frac{CF_1(1-D)}{r-g}$ starting from first principles.

8. A portfolio has 60% allocation to equities (β = 1.3) and 40% to bonds (β = 0.2). If climate risk adds a premium of 200 bp to equities and 50 bp to bonds, calculate the portfolio's climate risk premium.

9. Show that for small damages D and small growth rate g, the percentage value loss in a growing perpetuity approximately equals D. (Hint: Use Taylor expansion.)

10. Calculate the climate beta for an asset with:

    9. Correlation with climate damages: $\rho = 0.4$
    10. Asset volatility: $\sigma_i = 25\%$
    11. Climate damage volatility: $\sigma_D = 20\%$
    12. Market volatility: $\sigma_m = 18\%$

11. A project has uncertain cash flows: \$500 ± \$100 (uniform distribution) per year for 5 years. Climate damages are 10% ± 5% (uniform). Discount rate is 7%. Use Monte Carlo (1,000 simulations) to estimate the expected NPV and 90% confidence interval.

12. Prove that $ES_c \geq VaR_c$ for any loss distribution. Under what conditions does equality hold?

Advanced Problems (13-18)

13. **Climate-adjusted CAPM derivation:** Derive Equation 2.11 rigorously using the multi-factor asset pricing framework. Show all steps from the basic CAPM to the climate-augmented model.

14. **Time-varying climate risk premium: Model $rp_{climate}(t)$ as an increasing function of temperature: $rp(t) = \alpha \cdot T(t)^2$. Given $T(t) = 1.0 + 0.02t$ (°C) and $\alpha = 0.005$, calculate the present value of a perpetual cash flow of \$100 starting in year 10, using time-varying discount rates.**

15. **Non-linear damage functions:** An asset generates $1,000/year for 20 years. Damages follow $D(T) = 0.002T^2$. Temperature increases linearly from 1.2°C to 3.0°C over 20 years. Discount rate is 6%. Calculate the climate-adjusted NPV.

16. **Portfolio optimization with climate risk:** An investor allocates between two assets:

    13. Asset A: $E[R] = 10\%$, $\sigma = 15\%$, climate beta = 0.5
    14. Asset B: $E[R] = 8\%$, $\sigma = 10\%$, climate beta = 0.1
    15. Correlation: $\rho_{AB} = 0.3$
    16. Climate risk premium: 2%

    Find the minimum variance portfolio and the tangency portfolio (assuming $r_f = 3\%$).

17. **Stress testing:** A bank's loan portfolio has expected losses of $50M ($\sigma$ = $100M) under baseline climate. Under RCP8.5, damages increase by 50% and volatility doubles. Calculate the change in 99.9% VaR and ES.

18. **Real options under climate uncertainty:** A mining project requires $500M investment and generates $80M/year for 20 years. The company has the option to abandon after year 10 for salvage value of $200M. Climate damages are uncertain: 5% (prob 0.6) or 20% (prob 0.4) starting year 5. Discount rate is 10%. Should the company invest? What is the value of the abandonment option?

References

[1] Burke, M., Hsiang, S. M., & Miguel, E. (2015). "Global non-linear effect of temperature on economic production." *Nature*, 527, 235-239. https://doi.org/10.1038/nature15725

[2] Dietz, S., Bowen, A., Dixon, C., & Gradwell, P. (2016). "'Climate value at risk' of global financial assets." *Nature Climate Change*, 6(7), 676-679.

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Chapter 3: Economic Damage Functions

3.1 Linking Temperature to Economic Output

To translate physical climate change into financial impact, we must establish a mathematical link between climate variables and economic outcomes. The primary tool for this is the **economic damage function**, which relates changes in climate variables, most commonly temperature, to changes in economic output (e.g., Gross Domestic Product - GDP).

Definition 3.1 (Climate Damage Function): A climate damage function, D(T), is a mathematical expression that quantifies the fractional loss in economic output as a function of the global mean temperature anomaly, T.

$$GDP_{impacted} = GDP_{baseline} \cdot (1 - D(T)) \text{(Eq. 3.1)}$$

These functions are a critical component of Integrated Assessment Models (IAMs) and are essential for estimating the social cost of carbon and for valuing assets under different climate scenarios.

3.2 The Burke-Hsiang-Miguel Non-Linear Model

A significant body of recent econometric research has demonstrated that the relationship between temperature and economic productivity is fundamentally non-linear. The work by Burke, Hsiang, and Miguel (2015) provides a globally generalizable, empirically-derived functional form for this relationship [1].

3.2.1 Model Specification

The model specifies the growth rate of economic output for a country *i* in year *t* as a quadratic function of temperature:

$$\Delta y_{¿} = \beta_1 T_{¿} + \beta_2 T_{¿}^2 + \gamma P_{¿} + \mu P_{¿}^2 + \alpha_i + \delta_t + \varepsilon_{¿} \text{(Eq. 3.2)}$$

Where: - $\Delta y_{¿}$ is the first difference of the natural log of GDP per capita (i.e., the growth rate) - $T_{¿}$ is the annual average temperature - $P_{¿}$ represents precipitation variables - α_i are country-specific fixed effects - δ_t are year-specific fixed effects - β_1, β_2 are the key coefficients capturing the non-linear temperature effect

Important Note on Parameters: The original BHM (2015) paper does not report simple β_1 and β_2 coefficients because the model includes country and year fixed effects. The relationship is estimated from panel data regression. For illustrative purposes in this chapter, we use simplified coefficients that approximate the marginal effects reported in their Figure 2. **For rigorous financial modeling, practitioners should use the full BHM specification from their replication data** [1].

Illustrative Simplified Coefficients (for pedagogical use only): - $\beta_1 \approx 0.0127$ (positive effect of temperature on growth at low temperatures) - $\beta_2 \approx -0.0005$ (negative quadratic term, creating inverted-U shape)

Theorem 3.1 (Optimal Productivity Temperature)

Statement: Given the quadratic relationship for economic growth specified by Burke, Hsiang, and Miguel, there exists an optimal temperature, T_{opt}, at which economic productivity is maximized. This optimum is given by:

$$T_{opt} = \frac{-\beta_1}{2\beta_2} \text{(Eq. 3.3)}$$

Proof:

To find the temperature that maximizes the economic growth rate ($\Delta y_{¿}$), we take the first derivative of the growth equation with respect to temperature ($T_{¿}$) and set it to zero:

$$\frac{d(\Delta y_{¿})}{dT_{¿}} = \frac{d}{dT_{¿}}(\beta_1 T_{¿} + \beta_2 T_{¿}^2 + \ldots) = \beta_1 + 2\beta_2 T_{¿}$$

Setting the derivative to zero to find the extremum:

$$0 = \beta_1 + 2\beta_2 T_{¿}$$

$$-\beta_1 = 2\beta_2 T_{¿}$$

$$T_{¿} = \frac{-\beta_1}{2\beta_2}$$

To confirm this is a maximum, we check the second derivative:

$$\frac{d^2(\Delta y_{¿})}{dT_{¿}^2}=2\beta_2$$

Empirical estimates from Burke, Hsiang, and Miguel find that $\beta_1>0$ and $\beta_2<0$, which means the second derivative is negative. Therefore, the function is concave, and the derived temperature T_{opt} is a maximum. ∎

Empirical analysis places this optimal temperature at approximately **13°C** [1].

3.2.2 Marginal Effects

The marginal effect of temperature on growth at any temperature T is:

$$ME(T)=\beta_1+2\beta_2 T \text{(Eq. 3.4)}$$

At the optimal temperature (T = 13°C), ME(13) = 0. For T > 13°C, ME(T) < 0 (warming reduces growth). For T < 13°C, ME(T) > 0 (warming increases growth).

3.3 Integrated Assessment Models (IAMs): The DICE Model

Traditional Integrated Assessment Models, such as the Dynamic Integrated Climate-Economy (DICE) model developed by William Nordhaus, often use a simpler, calibrated damage function. The DICE model typically employs a quadratic function of the temperature anomaly.

DICE Model Damage Function:

$$D(T_t)=\frac{\pi_1 T_t+\pi_2 T_t^2}{1+\pi_1 T_t+\pi_2 T_t^2}\text{(Eq. 3.5)}$$

For the simplified version often used in practice:

$$D(T_t)=\pi_1 T_t+\pi_2 T_t^2\text{(Eq. 3.5a)}$$

Where: - $D(T_t)$ is the fractional loss of GDP - T_t is the global mean temperature increase above pre-industrial levels (in °C) - π_1 and π_2 are calibrated coefficients

In the DICE-2016R2 model [2]: - $\pi_1=0$ (no linear term) - $\pi_2=0.00236$

This implies that damages are zero at T=0 and increase quadratically with temperature. This formulation does not include an optimal temperature; any warming causes damage.

Table 3.1: DICE Model Damage Estimates

Temperature Anomaly (°C)	Damage (% of GDP)	Cumulative Effect Over 50 Years
1.0	0.24%	~12%
2.0	0.94%	~38%
3.0	2.12%	~67%
4.0	3.77%	~91%
5.0	5.90%	~115%

Source: Nordhaus (2017) [2].

3.4 Sectoral Damage Functions

Aggregate damage functions can be decomposed into sector-specific impacts:

$$D_{total}(T) = \sum_i w_i \cdot D_i(T) \text{(Eq. 3.6)}$$

where w_i is the weight (GDP share) of sector i.

Key Sectors: 1. **Agriculture:** $D_{ag}(T) = \alpha_{ag} T + \beta_{ag} T^2$ 2. **Infrastructure:** $D_{infra}(T) = \gamma \cdot P(extreme_{events} \vee T)$ 3. **Health:** $D_{health}(T) = \delta \cdot mortality(T) + \epsilon \cdot morbidity(T)$ 4. **Energy:** $D_{energy}(T) = \zeta \cdot cooling_{demand}(T) - \eta \cdot heating_{demand}(T)$

3.5 Mathematical Comparison of Damage Functions

Table 3.2: Comparison of Damage Function Approaches

Feature	Burke-Hsiang-Miguel (BHM) Model	DICE Model
Functional Form	Non-linear (quadratic) in temperature levels	Non-linear (quadratic) in temperature anomaly

Derivation	Empirically estimated from historical data	Calibrated based on survey of expert opinion
Optimal Temp.	Yes, at T ≈ 13°C. Countries cooler than this may benefit from initial warming	No. All warming is damaging
Impact Path	Affects the growth rate of the economy	Affects the level of economic output
Long-term Effect	Compounds over time (growth effect)	Constant percentage loss (level effect)

Theorem 3.2 (Growth vs. Level Effects)

Statement: A damage function that affects the growth rate leads to exponentially larger long-term damages compared to one that affects the level of output.

Proof:

Let g_0 be the baseline growth rate and d be the constant damage to growth rate.

Level effect (DICE-type):

$$GDP_t^{level} = GDP_0 ¿$$

Growth effect (BHM-type):

$$GDP_t^{growth} = GDP_0 ¿$$

The ratio of damages is:

$$GDP_0 ¿¿$$

Simplifying:

$$¿¿¿$$

As $t \rightarrow \infty$, if $d < g_0$, the numerator grows exponentially while the denominator grows linearly in d, so the ratio $\rightarrow \infty$. ∎

This demonstrates that growth effects compound dramatically over time.

3.6 Worked Examples

Example 3.1: Calculating GDP Impact with the BHM Model

Problem: A country has a current average temperature of 25°C. Climate models project a 2°C warming. Using the illustrative BHM model coefficients ($\beta_1 = 0.0127$, $\beta_2 = -0.0005$), calculate the percentage change in the economic growth rate.

Solution:

The change in growth rate is the difference between the growth function evaluated at the new and old temperatures.

$$\Delta Growth = (\beta_1 T_{new} + \beta_2 T_{new}^2) - (\beta_1 T_{old} + \beta_2 T_{old}^2)$$

where $T_{old} = 25\,°C$ and $T_{new} = 27\,°C$.

Calculate growth effect at T_{old}:

$$Effect_{old} = 0.0127(25) - 0.0005(25^2) = 0.3175 - 0.3125 = 0.0050$$

Calculate growth effect at T_{new}:

$$Effect_{new} = 0.0127(27) - 0.0005(27^2) = 0.3429 - 0.3645 = -0.0216$$

Calculate the change in the growth rate:

$$\Delta Growth = -0.0216 - 0.0050 = -0.0266$$

Answer: The economic growth rate is projected to decrease by **2.66 percentage points.** For a country with baseline growth of 3%, this would reduce it to 0.34%, a dramatic impact. ∎

Example 3.2: Comparing BHM and DICE Damages

Problem: Calculate the percentage GDP loss for a 3°C temperature increase using both the DICE-2016R2 damage function and by approximating the BHM impact.

Solution:

DICE Approach:

Using $D(T)=\pi_2 T^2$ with $\pi_2=0.00236$:

$$Damage=0.00236\times ¿$$

Answer (DICE): A **2.12%** loss in the level of GDP.

BHM Approach (Illustrative):

Assume a country is at the optimal temperature of 13°C and warms to 16°C. The change in the annual growth rate is:

$$\Delta Growth=(\beta_1(16)+\beta_2(16^2))-(\beta_1(13)+\beta_2(13^2))$$

$$¿(0.0127\times 16-0.0005\times 256)-(0.0127\times 13-0.0005\times 169)$$

$$¿(0.2032-0.128)-(0.1651-0.0845)$$

$$¿0.0752-0.0806=-0.0054$$

This is a **0.54%** reduction in the annual growth rate.

Over 50 years, the cumulative effect is:

$$GDP_{50}=GDP_0 ¿$$

If baseline growth $g=0.02$ (2%), then:

$$\frac{GDP_{50}^{baseline}}{GDP_{50}^{climate}}=¿¿$$

Answer (BHM): GDP would be **31.2% lower** than baseline after 50 years, compared to only 2.12% in the DICE model. This illustrates the dramatic difference between growth and level effects. ∎

Example 3.3: Optimal Temperature Calculation

Problem: Using the illustrative BHM coefficients, calculate the optimal temperature for economic productivity and verify it is a maximum.

Solution:

Using Eq. 3.3:

$$T_{opt} = \frac{-\beta_1}{2\beta_2} = \frac{-0.0127}{2(-0.0005)} = \frac{-0.0127}{-0.001} = 12.7\,°C$$

Verification that this is a maximum:

The second derivative is:

$$\frac{d^2(\Delta y)}{dT^2} = 2\beta_2 = 2(-0.0005) = -0.001 < 0$$

Since the second derivative is negative, the function is concave down, confirming this is a maximum.

Marginal effect at T = 12.7°C:

$$ME(12.7) = 0.0127 + 2(-0.0005)(12.7) = 0.0127 - 0.0127 = 0$$

Answer: The optimal temperature is **12.7°C** (approximately 13°C), which matches the empirical finding of Burke et al. (2015). At this temperature, the marginal effect of additional warming is zero. ∎

Example 3.4: Sectoral Damage Aggregation

Problem: An economy has three sectors with the following characteristics:

Sector	GDP Share	Damage Function at T=3°C
Agriculture	15%	$D_{ag}(3) = 0.08$ (8%)

Manufacturing	45%	$D_{mfg}(3)=0.02$ (2%)
Services	40%	$D_{svc}(3)=0.01$ (1%)

Calculate the aggregate damage to GDP at T = 3°C.

Solution:

Using Eq. 3.6:

$$D_{total}(3)=\sum_i w_i \cdot D_i(3)$$

$$\text{¿}0.15\times 0.08+0.45\times 0.02+0.40\times 0.01$$

$$\text{¿}0.012+0.009+0.004$$

$$\text{¿}0.025=2.5\%$$

Interpretation: Although agriculture faces 8% damages, its smaller share (15%) means the aggregate damage is only 2.5%. This demonstrates the importance of sectoral composition in determining overall climate vulnerability.

Answer: The aggregate damage to GDP at T = 3°C is **2.5%.** ∎

Example 3.5: Adaptation Cost-Benefit Analysis

Problem: A country faces projected climate damages of 5% of GDP (D = 0.05) at T = 4°C. An adaptation investment of 1% of GDP can reduce damages to 3% (D = 0.03). The country's GDP is $500 billion, and the discount rate is 5%. The adaptation investment must be made now, while benefits accrue over 30 years. Should the country invest in adaptation?

Solution:

Cost of adaptation:

$$C_{adapt}=0.01\times \$500\,B=\$5\,B$$

Annual benefit (damage reduction):

$$B_{annual} = (0.05 - 0.03) \times \$500B = 0.02 \times \$500B = \$10B$$

Present value of benefits over 30 years:

$$PV_{benefits} = B_{annual} \times 1 - ¿¿$$

$$¿ \$10B \times 1 - ¿¿$$

$$¿ \$10B \times \frac{1 - 0.2314}{0.05}$$

$$¿ \$10B \times 15.372 = \$153.72B$$

Net present value:

$$NPV = PV_{benefits} - C_{adapt} = \$153.72B - \$5B = \$148.72B$$

Benefit-cost ratio:

$$BCR = \frac{PV_{benefits}}{C_{adapt}} = \frac{\$153.72B}{\$5B} = 30.7$$

Answer: Yes, the country should invest in adaptation. The NPV is **$148.72 billion** with a benefit-cost ratio of **30.7:1**, indicating a highly favorable investment. ∎

Example 3.6: Tipping Point Modeling

Problem: A damage function includes a tipping point at T = 2.5°C, beyond which damages increase sharply:

$$D(T) = ¿$$

Calculate damages at T = 2°C, T = 2.5°C, and T = 4°C.

Solution:

At T = 2°C (below tipping point):

$$D(2) = 0.002 \times ¿$$

At T = 2.5°C (at tipping point):

$$D(2.5) = 0.002 \times ¿$$

At T = 4°C (beyond tipping point):

$$D(4) = 0.002 ¿$$

$$¿ 0.0125 + 0.01 ¿$$

$$¿ 0.0125 + 0.01 \times 2.25$$

$$¿ 0.0125 + 0.0225 = 0.035 = 3.5\%$$

Comparison: Without the tipping point, damages at T = 4°C would be:

$$D_{no_{tipping}}(4) = 0.002 \times ¿$$

Answer: Damages are **0.8%** at T=2°C, **1.25%** at T=2.5°C, and **3.5%** at T=4°C. The tipping point adds an additional **0.3%** damage at T=4°C compared to the smooth function. ∎

Example 3.7: Regional Heterogeneity in Damages

Problem: Two countries have different baseline temperatures: - Country A: T_0 = 10°C (cool climate) - Country B: T_0 = 20°C (warm climate)

Both experience 2°C warming. Using the BHM model, calculate the change in growth rate for each country.

Solution:

Country A (10°C → 12°C):

Initial effect:

$$Effect_A(10) = 0.0127(10) - 0.0005(10^2) = 0.127 - 0.05 = 0.077$$

Final effect:

$$Effect_A(12)=0.0127(12)-0.0005(12^2)=0.1524-0.072=0.0804$$

Change:

$$\Delta Growth_A=0.0804-0.077=+0.0034=+0.34\%$$

Country B (20°C → 22°C):

Initial effect:

$$Effect_B(20)=0.0127(20)-0.0005(20^2)=0.254-0.2=0.054$$

Final effect:

$$Effect_B(22)=0.0127(22)-0.0005(22^2)=0.2794-0.242=0.0374$$

Change:

$$\Delta Growth_B=0.0374-0.054=-0.0166=-1.66\%$$

Answer: Country A (cool climate) experiences a **+0.34%** increase in growth rate, while Country B (warm climate) suffers a **-1.66%** decrease. This demonstrates that climate change impacts are highly heterogeneous, with cool countries potentially benefiting while warm countries suffer. ∎

Example 3.8: Long-term Compounding of Growth Effects

Problem: A country with baseline GDP of $1 trillion and growth rate of 2% experiences a permanent 0.5% reduction in growth rate due to climate change. Calculate the GDP loss after 50 and 100 years.

Solution:

Baseline GDP trajectory:

$$GDP_t^{baseline}=GDP_0\text{¿}$$

Climate-impacted GDP trajectory:

$$GDP_t^{climate} = GDP_0 ¿$$

After 50 years:

$$GDP_{50}^{baseline} = \$1T \times ¿$$

$$GDP_{50}^{climate} = \$1T \times ¿$$

$$Loss_{50} = \$2.692T - \$2.105T = \$0.587T$$

$$\text{Percentage loss} = \frac{0.587}{2.692} = 21.8\%$$

After 100 years:

$$GDP_{100}^{baseline} = \$1T \times ¿$$

$$GDP_{100}^{climate} = \$1T \times ¿$$

$$Loss_{100} = \$7.245T - \$4.432T = \$2.813T$$

$$\text{Percentage loss} = \frac{2.813}{7.245} = 38.8\%$$

Answer: After 50 years, GDP is **21.8% lower** ($587 billion loss). After 100 years, GDP is **38.8% lower** ($2.813 trillion loss). This demonstrates the dramatic compounding effect of growth rate damages over time. ∎

3.7 Supplementary Problems

Basic Problems (1-5)

- Using the illustrative BHM coefficients ($\beta_1 = 0.0127$, $\beta_2 = -0.0005$), calculate the marginal effect of temperature on growth at T = 15°C, T = 20°C, and T = 25°C. Interpret the results.

- For the DICE-2016R2 model with $\pi_2 = 0.00236$, at what temperature anomaly does the GDP loss reach 10%? Solve for T.

- A country at T = 18°C experiences 1°C warming. Will its growth rate increase or decrease according to the BHM model? Calculate the exact change.
- Calculate the aggregate damage for an economy with two sectors: Agriculture (20% of GDP, 12% damage at T=4°C) and Services (80% of GDP, 2% damage at T=4°C).
- Verify that the second derivative of the BHM growth function is negative, confirming the optimal temperature is a maximum.

Intermediate Problems (6-11)

(f) Derive a formula for the difference in GDP level after N years between a baseline growth rate g and a climate-impacted growth rate (g - d), assuming the damage d is constant. Show that the ratio diverges exponentially.

(g) A country faces a choice between two adaptation strategies:

17. Strategy A: Invest $10B now, reduce damages from 6% to 3% for 40 years
18. Strategy B: Invest $5B now, reduce damages from 6% to 4% for 40 years

GDP is $800B, discount rate is 4%. Which strategy has higher NPV?

(h) Prove that for the DICE damage function $D(T) = \pi_2 T^2$, the marginal damage (dD/dT) increases linearly with temperature. Calculate the marginal damage at T = 2°C and T = 4°C.

(i) A damage function includes regional variation: $D(T, latitude) = \alpha T^2 \cdot ¿$ where $\alpha = 0.002$ and $\beta = 0.01$. Calculate damages at T = 3°C for latitudes 30°, 45°, and 60°.

(j) Compare the 100-year cumulative GDP loss for:

19. Level effect: D = 0.03 (constant 3% loss)
20. Growth effect: d = 0.003 (0.3% growth reduction)

Assume baseline GDP = $1T, g = 0.025.

(k) A tipping point model has: $D(T) = 0.001T^2$ for T ≤ 3°C, and $D(T) = 0.009 + 0.05(T - 3)$ for T > 3°C. Calculate the discontinuity in the derivative at T = 3°C.

Advanced Problems (12-15)

12. **Stochastic damage functions:** Assume temperature follows T(t) = 1 + 0.02t + 0.3W(t) where W(t) is a Wiener process. The damage function is $D(T)=0.002T^2$. Derive the expected damage E[D(T(t))] at t = 50 years. (Hint: Use Itô's lemma and the fact that $E[W^2(t)] = t$.)

13. **Optimal adaptation investment:** A country can invest amount I (as fraction of GDP) in adaptation, which reduces damages according to: $D_{adapted}(T, I)=D_0(T) \cdot e^{-\gamma I}$ where $\gamma=5$. If $D_0(T)=0.05$ and the investment cost is I, find the optimal I that minimizes total cost (damages + investment).

14. **Heterogeneous agents:** An economy has two regions with populations N_1 = 100M and N_2 = 50M, and per-capita damages $d_1(T)=0.001T^2$ and $d_2(T)=0.003T^2$. Derive the aggregate per-capita damage function $\acute{d}(T)$ and calculate it at T = 3°C.

15. **Non-linear tipping cascades:** A system has two tipping points:

    21. At T = 2°C: Ice sheet collapse adds 0.01 to damage coefficient
    22. At T = 3.5°C: Amazon dieback adds another 0.02

    Model this as: $D(T)=\pi(T) \cdot T^2$ where $\pi(T)$ is a step function. Calculate damages at T = 1.5°C, 2.5°C, and 4°C. Derive the expected damage if T is uniformly distributed on [2, 4].

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Chapter 4: Stochastic Processes for Climate and Finance

4.1 Introduction to Stochastic Differential Equations (SDEs)

Deterministic models, while useful, do not capture the inherent randomness and uncertainty present in both financial markets and climate systems. Stochastic processes, and specifically Stochastic Differential Equations (SDEs), provide a rigorous framework for modeling systems that evolve over time in a probabilistic manner.

An SDE models the evolution of a variable as the sum of a deterministic drift component and a stochastic diffusion component.

Definition 4.1 (General Form of an SDE): A standard one-dimensional SDE for a process X_t is given by:

$$d X_t = a(t, X_t) dt + b(t, X_t) d W_t \text{(Eq. 4.1)}$$

Where: - X_t is the stochastic process - $a(t, X_t)$ is the drift function, representing the deterministic trend - $b(t, X_t)$ is the diffusion function, representing the volatility or magnitude of the random fluctuations - $d W_t$ is a Wiener process (or Brownian motion), which has the properties: 1. $d W_t$ has a normal distribution with mean 0 and variance dt 2. For any two different time intervals, the corresponding increments $d W_t$ are independent

Properties of Wiener Process:

Theorem 4.1 (Wiener Process Properties)

A Wiener process W_t satisfies: 1. $W_0 = 0$ (starts at zero) 2. W_t has independent increments 3. $W_t - W_s \sim N(0, t-s)$ for $t > s$ 4. W_t has continuous paths

Proof: (Standard result from probability theory - see [1] for complete proof)

The key property for stochastic calculus is that ¿ in the mean-square sense, which leads to Itô's calculus.

4.2 Modeling Asset Prices with Geometric Brownian Motion (GBM)

The most common SDE used in finance is the Geometric Brownian Motion (GBM) model for stock prices. It assumes that the percentage returns of an asset are normally distributed.

Definition 4.2 (Geometric Brownian Motion): The SDE for an asset price S_t following a GBM is:

$$d S_t = \mu S_t dt + \sigma S_t d W_t \text{(Eq. 4.2)}$$

Where: - μ is the constant drift rate (expected return) - σ is the constant volatility of the asset

This equation states that the change in the stock price ($d S_t$) is composed of a deterministic part proportional to the current price ($\mu S_t dt$) and a stochastic part, also proportional to the current price ($\sigma S_t d W_t$).

Theorem 4.2 (Solution to GBM)

The solution to the GBM equation (Eq. 4.2) is:

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right] \text{(Eq. 4.3)}$$

Proof:

Let $Y_t = \ln(S_t)$. We will apply Itô's Lemma (Theorem 4.3 below) to find the SDE for Y_t.

For $f(S) = \ln(S)$: - $\frac{\partial f}{\partial S} = \frac{1}{S}$ - $\frac{\partial^2 f}{\partial S^2} = \frac{-1}{S^2}$

Applying Itô's Lemma:

$$d Y_t = \left[\mu S \cdot \frac{1}{S} + \frac{1}{2}\sigma^2 S^2 \cdot \left(\frac{-1}{S^2}\right)\right] dt + \sigma S \cdot \frac{1}{S} d W_t$$

$$d Y_t = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma d W_t$$

This is a simple Wiener process with drift. Integrating from 0 to t:

$$Y_t - Y_0 = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$$

$$\ln(S_t) - \ln(S_0) = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t$$

$$\ln\left(\frac{S_t}{S_0}\right) = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t$$

Exponentiating both sides:

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]$$

∎

4.3 Itô’s Lemma

Itô’s Lemma is the fundamental theorem of stochastic calculus. It is the stochastic equivalent of the chain rule and allows us to find the differential of a function of a stochastic process.

Theorem 4.3 (Itô’s Lemma)

Statement: Let X_t be a stochastic process that follows the SDE:

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

Let $f(t, x)$ be a twice-differentiable function. Then the process $Y_t = f(t, X_t)$ follows the SDE:

$$dY_t = ¿$$

Proof Outline:

The proof involves a Taylor series expansion of $f(t, X_t)$:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}¿$$

The key insight is that in stochastic calculus, we must keep terms up to order dt because: - ¿ (higher order infinitesimal) - ¿ (fundamental property of Wiener process) - $dt \cdot dW_t = 0$ (mixed terms vanish)

Substituting $dX_t = adt + bdW_t$:

¿

Keeping only terms of order dt:

$$df = \left[\frac{\partial f}{\partial t} + a \frac{\partial f}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2 f}{\partial x^2} \right] dt + b \frac{\partial f}{\partial x} dW_t$$

∎

Corollary 4.1 (Itô's Product Rule):

For two Itô processes X_t and Y_t:

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$$

where the last term $dX_t dY_t$ is computed using the multiplication table: - $dt \cdot dt = 0$ - $dt \cdot dW_t = 0$ - $dW_t \cdot dW_t = dt$

4.4 Climate-Driven SDEs

The standard GBM model can be extended to incorporate the financial impacts of climate change by making the drift and volatility parameters functions of a climate variable, such as temperature (T_t).

Definition 4.3 (Climate-Driven SDE for Asset Prices): A simple formulation for an asset price S_t impacted by climate change is:

$$dS_t = \mu(T_t) S_t dt + \sigma(T_t) S_t dW_t \text{(Eq. 4.5)}$$

Where: - $\mu(T_t)$ is the temperature-dependent drift. This can be directly linked to the economic damage functions from Chapter 3. For example, if the BHM model holds, the growth rate μ will be a quadratic function of temperature:

$$\mu(T) = \mu_0 + \beta_1 T + \beta_2 T^2$$

10. $\sigma(T_t)$ is the temperature-dependent volatility. There is evidence that climate change will increase economic volatility, making σ an increasing function of T_t:

$$\sigma(T) = \sigma_0 + \gamma T$$

4.4.1 Ornstein-Uhlenbeck Process for Temperature

Temperature anomalies can be modeled as mean-reverting processes:

Definition 4.4 (Ornstein-Uhlenbeck Process):

$$dT_t = \theta(\acute{T} - T_t)\,dt + \sigma_T\,dW_t \text{(Eq. 4.6)}$$

Where: - θ is the mean-reversion speed - $\acute{T}$ is the long-run mean temperature anomaly - σ_T is the temperature volatility

Theorem 4.4 (Solution to OU Process)

The solution to the OU process is:

$$T_t = T_0 e^{-\theta t} + \acute{T}(1 - e^{-\theta t}) + \sigma_T \int_0^t e^{-\theta(t-s)}\,dW_s \text{(Eq. 4.7)}$$

The expected value and variance are:

$$E[T_t] = T_0 e^{-\theta t} + \acute{T}(1 - e^{-\theta t})$$

$$Var[T_t] = \frac{\sigma_T^2}{2\theta}(1 - e^{-2\theta t})$$

As $t \to \infty$: $E[T_t] \to \acute{T}$ and $Var[T_t] \to \frac{\sigma_T^2}{2\theta}$ (stationary distribution).

4.5 Jump-Diffusion Models

GBM assumes continuous price movements. However, financial markets and climate systems can experience sudden, large shocks (e.g., a market crash or an extreme weather event). Jump-diffusion models extend the SDE framework to include these events.

Definition 4.5 (Merton Jump-Diffusion Model):

$$\frac{dS_t}{S_t} = (\mu - \lambda k)\,dt + \sigma\,dW_t + dJ_t \text{(Eq. 4.8)}$$

Where: - dJ_t is a compound Poisson process representing the jumps - λ is the jump intensity (average number of jumps per unit time) - $k=E[e^Y-1]$ where Y is the jump size (often $Y \sim N(\mu_J, \sigma_J^2)$)

Theorem 4.5 (Expected Return with Jumps)

For the Merton model, the expected instantaneous return is:

$$E\left[\frac{dS_t}{S_t}\right]=\mu dt$$

The term λk in the drift compensates for the expected jump size, ensuring the expected return remains μ.

Proof:

$$E[dJ_t]=E[\text{number of jumps}]\times E[\text{jump size}]=\lambda dt\times k$$

Therefore:

$$E\left[\frac{dS_t}{S_t}\right]=(\mu-\lambda k)dt+\lambda kdt=\mu dt$$

∎

4.5.1 Climate-Driven Jump Intensity

In a climate-finance context, the jump intensity λ can be modeled as a function of temperature, $\lambda(T_t)$, representing the increasing frequency of extreme weather events as the planet warms:

$$\lambda(T)=\lambda_0 e^{\alpha T} \text{ (Eq. 4.9)}$$

where $\alpha>0$ captures the exponential increase in extreme events with warming.

4.6 Multi-Dimensional SDEs and Correlation

Real-world applications often require modeling multiple correlated stochastic processes.

Definition 4.6 (Correlated Wiener Processes):

Two Wiener processes $W_t^{(1)}$ and $W_t^{(2)}$ with correlation ρ can be constructed as:

$$dW_t^{(1)} = dZ_t^{(1)}$$

$$dW_t^{(2)} = \rho dZ_t^{(1)} + \sqrt{1-\rho^2}\, dZ_t^{(2)}$$

where $Z_t^{(1)}$ and $Z_t^{(2)}$ are independent standard Wiener processes.

Verification:

$$E[dW_t^{(1)} dW_t^{(2)}] = E¿$$

✓

4.7 Worked Examples

Example 4.1: Simulating a Climate-Driven Asset Price Path

Problem: An asset's price follows $dS_t = \mu(T_t) S_t\, dt + \sigma S_t\, dW_t$. Let $S_0 = 100$, $\sigma = 0.20$. The drift is $\mu(T_t) = 0.08 - 0.01\, T_t^2$. The temperature anomaly T_t follows a simple path $T_t = 0.1t$. Simulate the asset price over one year (t=1) in a single time step.

Solution:

Discretize the SDE:

$$\Delta S \approx \mu(T) S \Delta t + \sigma S \sqrt{\Delta t} \cdot Z$$

where $Z \sim N(0,1)$.

Let $\Delta t = 1$. Then $T_1 = 0.1 \times 1 = 0.1$.

Calculate the drift at t=1:

$$\mu(0.1) = 0.08 - 0.01 ¿$$

Draw a random number from a standard normal distribution. Let $Z = -0.5$.

Calculate ΔS:

$$\Delta S \approx (0.0799 \times 100 \times 1) + (0.20 \times 100 \times \sqrt{1} \times (-0.5))$$

$$¿7.99-10=-2.01$$

Calculate the new asset price:

$$S_1=S_0+\Delta S=100-2.01=\$97.99$$

Answer: The simulated asset price after one year is **$97.99.** ∎

Example 4.2: Applying Itô's Lemma

Problem: Let an asset price S_t follow a GBM: $dS_t=\mu S_t dt+\sigma S_t dW_t$. Find the SDE for the process $Y_t=\ln(S_t)$.

Solution:

Let $f(S)=\ln(S)$. The derivatives are:

$$\frac{\partial f}{\partial S}=\frac{1}{S},\frac{\partial^2 f}{\partial S^2}=\frac{-1}{S^2}$$

Apply Itô's Lemma with $a=\mu S$ and $b=\sigma S$:

$$dY_t=\left[a\frac{\partial f}{\partial S}+\frac{1}{2}b^2\frac{\partial^2 f}{\partial S^2}\right]dt+b\frac{\partial f}{\partial S}dW_t$$

Substitute the derivatives and functions:

$$dY_t=¿$$

Simplify:

$$dY_t=\left[\mu-\frac{1}{2}\sigma^2\right]dt+\sigma dW_t$$

Answer: The log price follows $dY_t=(\mu-\frac{\sigma^2}{2})dt+\sigma dW_t$, a Wiener process with constant drift and diffusion. ∎

Example 4.3: Solving for Explicit Stock Price Formula

Problem: Solve the SDE for $Y_t = \ln(S_t)$ from Example 4.2 to find an explicit formula for S_t in terms of S_0, μ, σ, t, and W_t.

Solution:

From Example 4.2, we have:

$$dY_t = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma d W_t$$

This is a simple SDE with constant coefficients. Integrating from 0 to t:

$$Y_t - Y_0 = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$$

Since $Y_t = \ln(S_t)$ and $Y_0 = \ln(S_0)$:

$$\ln(S_t) - \ln(S_0) = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$$

$$\ln\left(\frac{S_t}{S_0}\right) = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$$

Exponentiating both sides:

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right]$$

Verification: Taking the differential of this expression using Itô's Lemma recovers the original GBM equation.

Answer: The explicit solution is $S_t = S_0 \exp\left[(\mu - \frac{\sigma^2}{2}) t + \sigma W_t\right]$. ∎

Example 4.4: Expected Value and Variance of GBM

Problem: For the GBM solution $S_t = S_0 \exp\left[(\mu - \frac{\sigma^2}{2}) t + \sigma W_t\right]$, calculate $E[S_t]$ and $Var[S_t]$.

Solution:

Expected Value:

Since $W_t \sim N(0,t)$, we have $\sigma W_t \sim N(0,\sigma^2 t)$.

For a log-normal random variable, if $X=e^Y$ where $Y \sim N(m,v^2)$, then:

$$E[X]=e^{m+v^2/2}$$

Here, $Y=(\mu-\frac{\sigma^2}{2})t+\sigma W_t$ with: - Mean: $m=(\mu-\frac{\sigma^2}{2})t$ - Variance: $v^2=\sigma^2 t$

Therefore:

$$E[S_t]=S_0 \exp\left[(\mu-\frac{\sigma^2}{2})t+\frac{\sigma^2 t}{2}\right]=S_0 e^{\mu t}$$

Variance:

For a log-normal variable, $Var[X]=E[X^2]-¿$.

$$E[S_t^2]=S_0^2 E[\exp(2Y)]=S_0^2 \exp\left[2m+2v^2\right]$$

$$¿\, S_0^2 \exp\left[2(\mu-\frac{\sigma^2}{2})t+2\sigma^2 t\right]$$

$$¿\, S_0^2 \exp\left[2\mu t+\sigma^2 t\right]$$

Therefore:

$$Var[S_t]=S_0^2 e^{2\mu t+\sigma^2 t}-S_0^2 e^{2\mu t}=S_0^2 e^{2\mu t}(e^{\sigma^2 t}-1)$$

Answer: $E[S_t]=S_0 e^{\mu t}$ and $Var[S_t]=S_0^2 e^{2\mu t}(e^{\sigma^2 t}-1)$. ∎

Example 4.5: Climate-Driven Volatility

Problem: Consider a climate-driven SDE where the volatility is a function of temperature: $\sigma(T_t)=0.2+0.05T_t$. If $T_t=2°C$, what is the new volatility? How would this affect the range of possible outcomes for the asset price over one year compared to constant volatility $\sigma=0.2$?

Solution:

New volatility:

$$\sigma(2)=0.2+0.05\times 2=0.2+0.1=0.3$$

Effect on outcomes:

For a GBM with $S_0=100$ and $\mu=0.08$, the standard deviation of $\ln(S_1)$ is:

Constant volatility ($\sigma=0.2$)**:**

$$\text{Std}[\ln(S_1)]=0.2\sqrt{1}=0.2$$

Climate-driven volatility ($\sigma=0.3$)**:**

$$\text{Std}[\ln(S_1)]=0.3\sqrt{1}=0.3$$

The 95% confidence interval for $\ln(S_1)$ is approximately $\pm 1.96\times\text{Std}$.

Constant volatility:

$$\ln(S_1)\in ¿$$

$$¿[4.605+0.08-0.02\pm 0.392]=[4.665\pm 0.392]$$

$$¿[4.273,5.057]$$

$$S_1\in[e^{4.273},e^{5.057}]=[\$71.6,\$157.2]$$

Climate-driven volatility:

$$\ln(S_1)\in ¿$$

$$¿[4.605+0.08-0.045\pm 0.588]=[4.640\pm 0.588]$$

$$¿[4.052,5.228]$$

$$S_1\in[e^{4.052},e^{5.228}]=[\$57.5,\$186.0]$$

Answer: The volatility increases from 0.2 to 0.3 (50% increase). This widens the 95% confidence interval from [$71.6, $157.2] to [$57.5, $186.0], representing significantly greater uncertainty in outcomes. ∎

Example 4.6: Jump-Diffusion with Climate-Driven Jump Intensity

Problem: In the Merton jump-diffusion model, the baseline jump intensity is $\lambda_0 = 0.1$ jumps/year. Due to climate change, the intensity increases to $\lambda(T) = \lambda_0 e^{0.2T}$. If temperature anomaly T = 3°C, calculate the new jump intensity and the expected number of jumps over 10 years. If the average jump size is $k = -0.15$ (15% drop), how does this affect the expected return term $(\mu - \lambda k)$?

Solution:

New jump intensity at T = 3°C:

$$\lambda(3) = 0.1 \times e^{0.2 \times 3} = 0.1 \times e^{0.6} = 0.1 \times 1.822 = 0.1822 \text{ jumps/year}$$

Expected number of jumps over 10 years:

$$E[\text{jumps}] = \lambda(3) \times 10 = 0.1822 \times 10 = 1.822 \text{ jumps}$$

Effect on expected return:

Baseline (T = 0):

$$\mu - \lambda_0 k = \mu - 0.1 \times (-0.15) = \mu + 0.015$$

Climate scenario (T = 3°C):

$$\mu - \lambda(3) k = \mu - 0.1822 \times (-0.15) = \mu + 0.0273$$

Change in drift:

$$\Delta(\text{drift}) = 0.0273 - 0.015 = 0.0123 = 1.23\%$$

Financial Intuition: The jump compensation term λk becomes more negative (since $k < 0$ and λ increases), which actually **increases** the drift. This seems counterintuitive, but it reflects the mathematical requirement that the expected return remains μ despite more frequent negative jumps. In reality, investors would demand a higher μ (risk premium) to compensate for increased jump risk.

Answer: Jump intensity increases from 0.1 to 0.182 jumps/year (82% increase). Expected jumps over 10 years: 1.82. The drift term increases by 1.23%, but this is a mathematical artifact——in practice, the required return μ would increase to compensate for higher jump risk. ∎

Example 4.7: Ornstein-Uhlenbeck Temperature Process

Problem: Temperature anomaly follows an OU process: $d T_t = 0.1(2 - T_t)dt + 0.3\, d W_t$ with $T_0 = 1\,°C$. Calculate the expected temperature and variance at t = 5 years and t = 50 years.

Solution:

From the OU process $d T_t = \theta(\acute{T} - T_t)dt + \sigma_T\, d W_t$, we identify: - $\theta = 0.1$ - $\acute{T} = 2\,°C$ - $\sigma_T = 0.3$ - $T_0 = 1\,°C$

Expected value:

$$E[T_t] = T_0 e^{-\theta t} + \acute{T}(1 - e^{-\theta t})$$

At t = 5:

$$E[T_5] = 1 \times e^{-0.1 \times 5} + 2(1 - e^{-0.5})$$

$$\text{¿}\, e^{-0.5} + 2(1 - e^{-0.5})$$

$$\text{¿}\, 0.6065 + 2(0.3935)$$

$$\text{¿}\, 0.6065 + 0.787 = 1.394\,°C$$

At t = 50:

$$E[T_{50}] = 1 \times e^{-5} + 2(1 - e^{-5})$$

$$\text{¿}\, 0.0067 + 2(0.9933)$$

$$\text{¿}0.0067+1.9866=1.993\,°C\approx 2\,°C$$

Variance:

$$Var[T_t]=\frac{\sigma_T^2}{2\theta}\left(1-e^{-2\theta t}\right)$$

At t = 5:

$$Var[T_5]=\text{¿¿}$$

$$\text{¿}\frac{0.09}{0.2}(1-0.3679)$$

$$\text{¿}0.45\times 0.6321=0.284$$

$$\text{Std}[T_5]=\sqrt{0.284}=0.533\,°C$$

At t = 50:

$$Var[T_{50}]=\frac{0.09}{0.2}\left(1-e^{-10}\right)$$

$$\text{¿}0.45(1-0.000045)$$

$$\text{¿}0.45\times 0.99996=0.450$$

$$\text{Std}[T_{50}]=\sqrt{0.450}=0.671\,°C$$

Answer: At t=5 years: $E[T_5]$ = 1.39°C, $Std[T_5]$ = 0.53°C. At t=50 years: $E[T_{50}]$ ≈ 2.0°C (converged to long-run mean), $Std[T_{50}]$ = 0.67°C (converged to stationary variance). ∎

Example 4.8: Simulating Correlated Asset and Temperature Processes

Problem: An asset price and temperature are correlated with $\rho=0.6$. The asset follows $dS_t=0.08S_t\,dt+0.25S_t\,dW_t^S$ and temperature follows $dT_t=0.05\,dt+0.2\,dW_t^T$. Construct the correlated Wiener processes and simulate one time step ($\Delta t=1$) starting from $S_0=100$, $T_0=1\,°C$. Use random draws $Z_1=0.5$, $Z_2=-0.3$.

Solution:

Construct correlated Wiener processes:

$$dW_t^S = dZ_1$$

$$dW_t^T = \rho dZ_1 + \sqrt{1-\rho^2}\, dZ_2$$

With $\rho = 0.6$:

$$dW_t^T = 0.6\, dZ_1 + \sqrt{1-0.36}\, dZ_2 = 0.6\, dZ_1 + 0.8\, dZ_2$$

Simulate asset price:

$$\Delta S = 0.08 \times 100 \times 1 + 0.25 \times 100 \times \sqrt{1} \times 0.5$$

$$¿ 8 + 12.5 = 20.5$$

$$S_1 = 100 + 20.5 = \$120.5$$

Simulate temperature:

$$\Delta T = 0.05 \times 1 + 0.2 \times \sqrt{1} \times (0.6 \times 0.5 + 0.8 \times (-0.3))$$

$$¿ 0.05 + 0.2 \times (0.3 - 0.24)$$

$$¿ 0.05 + 0.2 \times 0.06$$

$$¿ 0.05 + 0.012 = 0.062$$

$$T_1 = 1 + 0.062 = 1.062\,°C$$

Verification of correlation: The increments are $\Delta W^S = 0.5$ and $\Delta W^T = 0.06$. The correlation is:

$$\rho = \frac{E[\Delta W^S \Delta W^T]}{\sqrt{Var[\Delta W^S] Var[\Delta W^T]}} = \frac{0.6 \times 1}{1} = 0.6$$

✓

Answer: After one year: $S_1 = \$120.5$, $T_1 = 1.062\,°C$. The positive correlation means both tend to move together (in this case, both increased). ∎

Example 4.9: Itô's Product Rule Application

Problem: Two assets follow GBMs: $d S_t^{(1)} = \mu_1 S_t^{(1)} dt + \sigma_1 S_t^{(1)} d W_t^{(1)}$ and $d S_t^{(2)} = \mu_2 S_t^{(2)} dt + \sigma_2 S_t^{(2)} d W_t^{(2)}$ where the Wiener processes are independent. Find the SDE for the product $P_t = S_t^{(1)} S_t^{(2)}$.

Solution:

Using Itô's product rule:

$$d\left(S_t^{(1)} S_t^{(2)}\right) = S_t^{(1)} d S_t^{(2)} + S_t^{(2)} d S_t^{(1)} + d S_t^{(1)} d S_t^{(2)}$$

Calculate each term:

First term:

$$S_t^{(1)} d S_t^{(2)} = S_t^{(1)} \left[\mu_2 S_t^{(2)} dt + \sigma_2 S_t^{(2)} d W_t^{(2)}\right]$$

$$\text{¿}\, \mu_2 S_t^{(1)} S_t^{(2)} dt + \sigma_2 S_t^{(1)} S_t^{(2)} d W_t^{(2)}$$

Second term:

$$S_t^{(2)} d S_t^{(1)} = S_t^{(2)} \left[\mu_1 S_t^{(1)} dt + \sigma_1 S_t^{(1)} d W_t^{(1)}\right]$$

$$\text{¿}\, \mu_1 S_t^{(1)} S_t^{(2)} dt + \sigma_1 S_t^{(1)} S_t^{(2)} d W_t^{(1)}$$

Third term (quadratic variation):

$$d S_t^{(1)} d S_t^{(2)} = \left[\mu_1 S_t^{(1)} dt + \sigma_1 S_t^{(1)} d W_t^{(1)}\right]\left[\mu_2 S_t^{(2)} dt + \sigma_2 S_t^{(2)} d W_t^{(2)}\right]$$

Using $dt \cdot dt = 0$, $dt \cdot dW = 0$, and $d W_t^{(1)} d W_t^{(2)} = 0$ (independent):

$$\text{¿}\, 0$$

Combining all terms:

$$d P_t = (\mu_1 + \mu_2) P_t dt + \sigma_1 P_t d W_t^{(1)} + \sigma_2 P_t d W_t^{(2)}$$

Answer: The product follows $d P_t = (\mu_1 + \mu_2) P_t dt + \sigma_1 P_t d W_t^{(1)} + \sigma_2 P_t d W_t^{(2)}$. The drift is the sum of individual drifts, and the diffusion has two independent components. ∎

Example 4.10: Calibrating GBM from Historical Data

Problem: Historical monthly stock prices over 2 years (24 months) show: - Average monthly return: $\acute{r}=0.008$ (0.8%) - Standard deviation of monthly returns: $s=0.05$ (5%)

Estimate the annual drift μ and volatility σ parameters for a GBM model.

Solution:

For GBM, the log returns are:

$$r_t=\ln\left(\frac{S_{t+\Delta t}}{S_t}\right)\sim N\left[\left(\mu-\frac{\sigma^2}{2}\right)\Delta t, \sigma^2\Delta t\right]$$

From the data (monthly, so $\Delta t=1/12$):

Mean of log returns:

$$E[r_t]=\left(\mu-\frac{\sigma^2}{2}\right)\times\frac{1}{12}=\acute{r}=0.008$$

Variance of log returns:

$$Var[r_t]=\sigma^2\times\frac{1}{12}=s^2=¿$$

From the variance equation:

$$\sigma^2=0.0025\times 12=0.03$$

$$\sigma=\sqrt{0.03}=0.1732=17.32\%$$

From the mean equation:

$$\mu-\frac{\sigma^2}{2}=0.008\times 12=0.096$$

$$\mu=0.096+\frac{0.03}{2}=0.096+0.015=0.111=11.1\%$$

Answer: The calibrated parameters are $\mu=11.1\%$ (annual drift) and $\sigma=17.32\%$ (annual volatility).

∎

4.8 Supplementary Problems

Basic Problems (1-6)

- For a Wiener process W_t, calculate $E[W_5]$, $Var[W_5]$, and $P(W_5>1)$.
- An asset follows $dS_t=0.10S_t dt+0.30S_t dW_t$ with $S_0=50$. Using the explicit solution formula, write the expression for S_t at any time t.
- Verify that ¿ by showing that E¿ and Var¿ (in the limit).
- For the OU process $dT_t=0.2(1.5-T_t)dt+0.25dW_t$, what is the long-run mean and long-run variance?
- In a jump-diffusion model with $\lambda=0.2$ jumps/year and average jump size $k=-0.10$, what is the expected number of jumps over 5 years? What is the total expected loss from jumps?
- Two independent Wiener processes $W_t^{(1)}$ and $W_t^{(2)}$ are combined: $W_t=0.6W_t^{(1)}+0.8W_t^{(2)}$. Show that W_t is also a Wiener process by verifying $Var[W_t]=t$.

Intermediate Problems (7-12)

7. Apply Itô's Lemma to find the SDE for $Y_t=S_t^2$ where S_t follows a GBM: $dS_t=\mu S_t dt+\sigma S_t dW_t$.
8. Solve the OU process $dT_t=\theta(\acute{T}-T_t)dt+\sigma_T dW_t$ explicitly by using the integrating factor method. Verify the solution given in Eq. 4.7.
9. For a GBM with $\mu=0.12$ and $\sigma=0.25$, calculate the probability that the stock price doubles ($S_t=2S_0$) within 5 years. (Hint: Use the log-normal distribution.)
10. Construct two correlated Wiener processes with $\rho=-0.5$ using independent standard Wiener processes $Z_t^{(1)}$ and $Z_t^{(2)}$. Verify the correlation.
11. A climate-driven asset has drift $\mu(T)=0.10-0.02T^2$ and volatility $\sigma(T)=0.20+0.03T$. If temperature increases from 1°C to 3°C, calculate the change in expected return and volatility.

12. For the Merton jump-diffusion model with $\mu = 0.08$, $\sigma = 0.20$, $\lambda = 0.15$, and jump sizes $Y \sim N(-0.05, 0.10^2)$, calculate the parameter $k = E[e^Y - 1]$ and the compensated drift $\mu - \lambda k$.

Advanced Problems (13-18)

13. **Girsanov's Theorem Application:** Under the risk-neutral measure, the drift of a GBM changes from μ to r (risk-free rate). Derive the Radon-Nikodym derivative for this measure change and show how it affects the Wiener process.

14. **Multi-dimensional Itô:** Two assets follow correlated GBMs with correlation ρ. Derive the SDE for the portfolio $V_t = w_1 S_t^{(1)} + w_2 S_t^{(2)}$ and find the portfolio volatility.

15. **Variance of OU process:** Derive the variance formula $Var[T_t] = \frac{\sigma_T^2}{2\theta}(1 - e^{-2\theta t})$ by solving the variance differential equation.

16. **Jump-diffusion option pricing:** For a European call option on an asset following Merton's jump-diffusion model, the price is a weighted sum of Black-Scholes prices. Derive the first term of this series (corresponding to zero jumps).

17. **Climate tipping point SDE:** Model a climate variable with a tipping point using: $dX_t = \theta(X_t)(X_{crit} - X_t)dt + \sigma dW_t$ where $\theta(X) = \theta_0$ for $X < X_{crit}$ and $\theta(X) = -\theta_0$ for $X > X_{crit}$. Analyze the stability of this system.

18. **Calibration with jumps:** Given historical data showing both continuous volatility and occasional large drops, develop a maximum likelihood estimator for the parameters $(\mu, \sigma, \lambda, \mu_J, \sigma_J)$ of the Merton model.

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Chapter 5: Monte Carlo Simulation for Risk Quantification

5.1 Mathematical Principles of Monte Carlo Methods

Monte Carlo methods are a broad class of computational algorithms that rely on repeated random sampling to obtain numerical results. The underlying principle is to use randomness to solve problems that might be deterministic in principle but are too complex to solve analytically. The core idea is based on the Law of Large Numbers.

Theorem 5.1 (Law of Large Numbers)

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables with a finite expected value $E[X]=\mu$. Then the sample mean, $\acute{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, converges to the expected value μ as $n \to \infty$:

$$\lim_{n \to \infty} \acute{X}_n = \mu \, (\text{almost surely}) \, (\text{Eq. 5.1})$$

Proof: (Standard result from probability theory - see [1])

Theorem 5.2 (Central Limit Theorem)

Under the same conditions as Theorem 5.1, and assuming finite variance σ^2, the distribution of the sample mean converges to a normal distribution:

$$\sqrt{n}(\bar{Y}_n - \mu) \to_D N(0, \sigma^2) \quad (\text{Eq. 5.2})$$

This implies the standard error of the Monte Carlo estimate is:

$$SE(\acute{X}_n) = \frac{\sigma}{\sqrt{n}} \, (\text{Eq. 5.3})$$

Corollary 5.1 (Convergence Rate):

The error in a Monte Carlo estimate decreases at rate $O(1/\sqrt{n})$. To halve the error, we must quadruple the number of simulations.

In the context of financial risk, if we can generate a large number of random scenarios (or paths) for the evolution of a portfolio's value under climate change, the average outcome of these scenarios will converge to the true expected value. More importantly, the distribution of these simulated outcomes can be used to estimate risk metrics like VaR and ES.

5.2 Algorithm for Climate Value-at-Risk (Climate VaR) Calculation

Climate VaR is an estimate of the potential loss in portfolio value due to climate change at a given confidence level over a specific time horizon. A Monte Carlo approach to calculating Climate VaR involves a multi-step process that integrates climate science, economics, and finance.

Algorithm 5.1: Monte Carlo Climate VaR

Objective: To calculate the α-percentile loss on a financial portfolio due to climate change over a horizon T.

Step 1: Scenario Generation (Climate Module)

For $i=1$ to N (where N is the number of simulations):

1a. Sample from the probability distributions of key climate parameters: - Radiative forcing: $F_i \sim N(\mu_F, \sigma_F^2)$ (e.g., from CMIP6 estimates) - Climate sensitivity parameter: $\lambda_i \sim N(\mu_\lambda, \sigma_\lambda^2)$

1b. For each sampled set of parameters, generate a future temperature path, $T_i(t)$, from $t=0$ to T. This can be done using: - Simplified climate model: $T_i(t) = F_i/\lambda_i \cdot (1-e^{-t/\tau})$ - Or by sampling from a pre-existing ensemble of GCM runs

Step 2: Impact Calculation (Economic Module)

For each temperature path $T_i(t)$:

2a. Apply a stochastic economic damage function, $D(T_i(t), \varepsilon_i)$, where ε_i is a random shock term, to translate the temperature path into a path of economic impacts (e.g., GDP growth rate shocks):

$$g_i(t) = g_0 - D(T_i(t), \varepsilon_i)$$

2b. Propagate these economic impacts through an economic model to generate paths for relevant macroeconomic variables (e.g., interest rates, market indices).

Step 3: Valuation (Finance Module)

For each macroeconomic path:

3a. Re-value the assets in the portfolio to determine the terminal portfolio value, $V_i(T)$:

$$V_i(T) = \sum_{j=1}^{m} w_j \cdot S_{ij}(T)$$

where w_j is the weight of asset j and $S_{ij}(T)$ is its value in scenario i.

3b. Calculate the portfolio loss for the i-th simulation:

$$L_i = V_{baseline} - V_i(T)$$

where $V_{baseline}$ is the expected portfolio value in a world without climate change.

Step 4: Risk Aggregation

4a. Collect the N simulated losses to form a loss distribution, $\{L_1, L_2, \ldots, L_N\}$.

4b. Sort the loss distribution in ascending order: $L_{(1)} \leq L_{(2)} \leq \cdots \leq L_{(N)}$.

4c. The Climate VaR at confidence level α is the value at the $\lceil N \cdot \alpha \rceil$-th position in the sorted loss distribution:

$$\text{VaR}_\alpha = L_{(\lceil N \cdot \alpha \rceil)} \text{(Eq. 5.4)}$$

4d. The Expected Shortfall (ES) at confidence level α is:

$$\text{ES}_\alpha = \frac{1}{N(1-\alpha)} \sum_{i : L_i > \text{VaR}_\alpha} L_i \text{(Eq. 5.5)}$$

5.3 Variance Reduction Techniques

The standard Monte Carlo method has slow convergence ($O(1/\sqrt{n})$). Variance reduction techniques can significantly improve efficiency.

5.3.1 Antithetic Variates

Definition 5.1 (Antithetic Variates):

For each random draw $Z \sim N(0,1)$, also simulate using $-Z$. This creates negative correlation between pairs, reducing variance.

Theorem 5.3 (Variance Reduction from Antithetic Variates):

For a function f that is monotonic in Z, the variance of the antithetic estimator is:

$$Var\left[\frac{f(Z)+f(-Z)}{2}\right] \leq Var[f(Z)] \text{(Eq. 5.6)}$$

5.3.2 Control Variates

Definition 5.2 (Control Variates):

Use a correlated variable with known expectation to reduce variance. If we want to estimate $E[X]$ and we know $E[Y]$:

$$\widehat{X}_{CV} = \acute{X} - \beta(\acute{Y} - E[Y]) \text{(Eq. 5.7)}$$

where β is chosen to minimize variance (optimal: $\beta^{¿} = Cov[X, Y] / Var[Y]$).

5.3.3 Importance Sampling

Definition 5.3 (Importance Sampling):

Sample from a different distribution $g(x)$ instead of the target $f(x)$, and reweight:

$$E_f[h(X)] = E_g\left[h(X)\frac{f(X)}{g(X)}\right] \text{(Eq. 5.8)}$$

This is particularly useful for rare events (e.g., extreme climate scenarios).

5.4 Propagating Uncertainty

A key strength of the Monte Carlo framework is its ability to formally propagate uncertainty through the entire modeling chain. The uncertainty in the final loss distribution is a composite of uncertainties from each stage:

(a) Climate Uncertainty: Uncertainty in radiative forcing, climate sensitivity, and the internal variability of the climate system.

(b) Economic Uncertainty: Uncertainty in the parameters of the damage function (e.g., the β coefficients in the BHM model) and shocks to economic growth.

(c) Financial Uncertainty: Uncertainty in asset-specific responses to macroeconomic shocks (i.e., uncertainty in asset betas).

By sampling from the probability distributions of the parameters at each stage, the Monte Carlo simulation produces a final loss distribution that reflects the combined effect of all these underlying uncertainties.

5.5 Worked Examples

Example 5.1: Complete Numerical Example (10 Scenarios)

Problem: Let's perform a simplified 10-scenario Monte Carlo simulation for a single asset.

Given: - **Asset:** A perpetual claim on a dividend stream. Current value (baseline) = $1000 - **Climate Model:** $\Delta T = F/\lambda$. We assume $F \sim N(3.0, 0.5^2)$ W/m² and $\lambda \sim N(1.2, 0.2^2)$ - **Damage Function:** Loss = $0.02 \times ¿$. This is a level-impact model for simplicity - **Objective:** Calculate the 90% Climate VaR

Solution:

Scenario (i)	Sampled F	Sampled λ	$\Delta T = F/\lambda$	Loss = $0.02 \times \Delta T^2$	Asset Value (V_i)	Loss (L_i)
1	3.2	1.1	2.91	16.9%	$831.00	$169.00
2	2.8	1.3	2.15	9.2%	$908.00	$92.00
3	3.5	1.0	3.50	24.5%	$755.00	$245.00
4	2.5	1.4	1.79	6.4%	$936.00	$64.00
5	3.8	1.2	3.17	20.1%	$799.00	$201.00
6	2.9	1.5	1.93	7.4%	$926.00	$74.00

7	3.1	0.9	3.44	23.7%	$763.00	$237.00
8	2.2	1.1	2.00	8.0%	$920.00	$80.00
9	4.0	1.3	3.08	19.0%	$810.00	$190.00
10	3.3	1.0	3.30	21.8%	$782.00	$218.00

Risk Aggregation:

- **Loss Distribution:** {$169, $92, $245, $64, $201, $74, $237, $80, $190, $218}
- **Sorted Losses:** {$64, $74, $80, $92, $169, $190, $201, $218, $237, $245}
- **VaR Calculation:** The 90% VaR is the 9th value ($N \times \alpha = 10 \times 0.9 = 9$) in the sorted list.

Answer: The 90% Climate VaR is **$237.** There is a 10% chance that the climate-related loss on the asset will exceed $237. ∎

Example 5.2: Calculating Expected Shortfall

Problem: From the case study data in Example 5.1, calculate the 90% Expected Shortfall (ES).

Solution:

ES is the average of all losses greater than or equal to the VaR.

From Example 5.1, VaR_{90} = $237.

Losses ≥ $237: {$237, $245}

$$ES_{90} = \frac{237 + 245}{2} = \frac{482}{2} = \$241$$

Alternative calculation (more conservative):

Some definitions use losses strictly greater than VaR:

Losses > $237: {$245}

$$ES_{90} = \$245$$

Answer: The 90% Expected Shortfall is **$241** (average of tail losses including VaR) or **$245** (average of losses strictly exceeding VaR). The first definition is more common. ∎

Example 5.3: Sensitivity to Damage Function

Problem: How would the Climate VaR change if the damage function was $D(T)=0.01\times T^2$ instead of $0.02\times T^2$? Recalculate the loss for scenario 1 and estimate the new 90% VaR.

Solution:

Scenario 1 with new damage function: - $\Delta T=2.91°C$ (unchanged) - Loss = $0.01\times ¿$ - Asset Value = $1000\times(1-0.0847)=\$915.30$ - Loss = $1000 - $915.30 = 84.70

The original loss was $169, so the new loss is approximately half.

Scaling all losses:

Since the damage function is halved, all losses will be approximately halved:

Original sorted losses: {$64, $74, $80, $92, $169, $190, $201, $218, $237, $245}

New sorted losses (approximate): {$32, $37, $40, $46, $84.50, $95, $100.50, $109, $118.50, $122.50}

New 90% VaR: Approximately **$118.50** (9th value).

Answer: With the halved damage function, the 90% Climate VaR decreases from $237 to approximately $118.50, a reduction of 50%. This demonstrates the high sensitivity of risk metrics to damage function parameters. ∎

Example 5.4: Convergence Analysis

Problem: A Monte Carlo simulation estimates $E[L]=150$ with standard deviation $\sigma=60$ using $N=100$ scenarios. Calculate the 95% confidence interval for the true mean. How many scenarios are needed to reduce the confidence interval width to ±$5?

Solution:

Standard error:

$$SE = \frac{\sigma}{\sqrt{N}} = \frac{60}{\sqrt{100}} = \frac{60}{10} = 6$$

95% confidence interval:

$$CI = \acute{L} \pm 1.96 \times SE = 150 \pm 1.96 \times 6 = 150 \pm 11.76 = [138.24, 161.76]$$

Required N for CI width = ±$5:

We need:

$$1.96 \times \frac{\sigma}{\sqrt{N}} = 5$$

$$\frac{60}{\sqrt{N}} = \frac{5}{1.96} = 2.551$$

$$\sqrt{N} = \frac{60}{2.551} = 23.52$$

$$N = ¿$$

Answer: Current 95% CI is [$138.24, $161.76]. To achieve CI width of ±$5, we need **554 scenarios** (5.5× increase). ∎

Example 5.5: Antithetic Variates Application

Problem: Estimate $E[e^Z]$ where $Z \sim N(0,1)$ using (a) standard Monte Carlo with 4 samples, and (b) antithetic variates with 2 pairs. Use random draws: $Z_1 = 0.5$, $Z_2 = -1.2$.

Solution:

(a) Standard Monte Carlo (4 independent samples):

Suppose we draw: $Z_1 = 0.5$, $Z_2 = -1.2$, $Z_3 = 0.8$, $Z_4 = -0.3$

$$\hat{E}_{std} = \frac{1}{4}\left(e^{0.5} + e^{-1.2} + e^{0.8} + e^{-0.3}\right)$$

$$¿\frac{1}{4}(1.649+0.301+2.226+0.741)=\frac{4.917}{4}=1.229$$

(b) Antithetic variates (2 pairs):

Pair 1: $Z_1=0.5,\ -Z_1=-0.5$ Pair 2: $Z_2=-1.2,\ -Z_2=1.2$

$$\hat{E}_{ant}=\frac{1}{4}\left(e^{0.5}+e^{-0.5}+e^{-1.2}+e^{1.2}\right)$$

$$¿\frac{1}{4}(1.649+0.606+0.301+3.320)=\frac{5.876}{4}=1.469$$

True value:

For $Z\sim N(0,1)$, $E[e^Z]=e^{1/2}=e^{0.5}=1.649$ (log-normal property).

Comparison: - Standard MC error: $¿1.229-1.649\vee¿0.420$ - Antithetic error: $¿1.469-1.649\vee¿0.180$

Answer: Antithetic variates reduced the error by 57% in this example. The variance reduction comes from the negative correlation between $f(Z)$ and $f(-Z)$ when f is monotonic. ∎

Example 5.6: Importance Sampling for Rare Events

Problem: Estimate $P(L>500)$ where $L=100e^Z$ and $Z\sim N(0,1)$. This is a rare event. Compare standard Monte Carlo (1000 samples) with importance sampling using $Z\sim N(2,1)$.

Solution:

Event of interest:

$$L>500\Rightarrow 100e^Z>500\Rightarrow e^Z>5\Rightarrow Z>\ln(5)=1.609$$

Under $N(0,1)$: $P(Z>1.609)=1-\Phi(1.609)=1-0.9463=0.0537=5.37\%$

(a) Standard Monte Carlo:

With 1000 samples from $N(0,1)$, we expect about $1000\times0.0537=53.7$ samples with $Z>1.609$.

Estimated probability: $\hat{p}_{std}=\dfrac{\text{count}(Z>1.609)}{1000}$

Standard error: $SE=\sqrt{\frac{p(1-p)}{n}}=\sqrt{\frac{0.0537\times 0.9463}{1000}}=0.0071=0.71\%$

(b) Importance Sampling from $N(2,1)$**:**

Sample $Z^{¿}\sim N(2,1)$ and reweight:

$$\hat{p}_{IS}=\frac{1}{n}\sum_{i=1}^{n}1(Z_i^{¿}>1.609)\times\frac{\phi(Z_i^{¿};0,1)}{\phi(Z_i^{¿};2,1)}$$

where $\phi(\cdot;\mu,\sigma^2)$ is the normal PDF.

The likelihood ratio is:

$$\frac{\phi(z;0,1)}{\phi(z;2,1)}=\exp¿$$

Under $N(2,1)$, $P(Z^{¿}>1.609)=1-\Phi(1.609-2)=1-\Phi(-0.391)=\Phi(0.391)=0.652=65.2\%$

So we expect about 652 samples in the region of interest (vs. 54 for standard MC).

Answer: Importance sampling dramatically increases the number of samples in the tail region (652 vs. 54), reducing the standard error by approximately $\sqrt{652/54}=3.5$ times. For rare event estimation, this is a critical improvement. ∎

Example 5.7: Multi-Asset Portfolio VaR

Problem: A portfolio contains two assets with weights $w_1=0.6$, $w_2=0.4$. Initial values: $S_1(0)=100$, $S_2(0)=150$. They follow correlated GBMs with $\mu_1=0.08$, $\mu_2=0.10$, $\sigma_1=0.20$, $\sigma_2=0.25$, $\rho=0.5$. Calculate the 1-year 95% VaR using 5 Monte Carlo scenarios. Use random draws: $(Z_1^{(1)},Z_2^{(1)})=(0.5,0.3)$, $(Z_1^{(2)},Z_2^{(2)})=(-0.8,1.2)$, $(Z_1^{(3)},Z_2^{(3)})=(1.5,-0.5)$, $(Z_1^{(4)},Z_2^{(4)})=(-1.2,-0.9)$, $(Z_1^{(5)},Z_2^{(5)})=(0.2,0.7)$.

Solution:

Initial portfolio value:

$$V_0=0.6\times 100+0.4\times 150=60+60=\$120$$

Construct correlated Wiener increments:

$$W_1 = Z_1$$

$$W_2 = \rho Z_1 + \sqrt{1-\rho^2} Z_2 = 0.5 Z_1 + 0.866 Z_2$$

Asset price formula:

$$S_i(1) = S_i(0) \exp\left[(\mu_i - \frac{\sigma_i^2}{2}) + \sigma_i W_i \right]$$

Scenario 1: $(Z_1, Z_2) = (0.5, 0.3)$ - $W_1 = 0.5$, $W_2 = 0.5(0.5) + 0.866(0.3) = 0.25 + 0.260 = 0.510$ - $S_1(1) = 100 \exp[(0.08 - 0.02) + 0.20(0.5)] = 100 \exp[0.16] = 117.35$ - $S_2(1) = 150 \exp[(0.10 - 0.03125) + 0.25(0.510)] = 150 \exp[0.1963] = 182.66$ - $V_1 = 0.6(117.35) + 0.4(182.66) = 70.41 + 73.06 = \$143.47$ - Loss: $120 - 143.47 = -\$23.47$ (gain)

Scenario 2: $(Z_1, Z_2) = (-0.8, 1.2)$ - $W_1 = -0.8$, $W_2 = 0.5(-0.8) + 0.866(1.2) = -0.4 + 1.039 = 0.639$ - $S_1(1) = 100 \exp[0.06 - 0.16] = 100 \exp[-0.10] = 90.48$ - $S_2(1) = 150 \exp[0.06875 + 0.1598] = 150 \exp[0.2285] = 188.73$ - $V_2 = 0.6(90.48) + 0.4(188.73) = 54.29 + 75.49 = \$129.78$ - Loss: $120 - 129.78 = -\$9.78$ (gain)

Scenario 3: $(Z_1, Z_2) = (1.5, -0.5)$ - $W_1 = 1.5$, $W_2 = 0.75 - 0.433 = 0.317$ - $S_1(1) = 100 \exp[0.06 + 0.30] = 143.33$ - $S_2(1) = 150 \exp[0.06875 + 0.0793] = 173.83$ - $V_3 = 0.6(143.33) + 0.4(173.83) = \$155.53$ - Loss: $-\$35.53$ (gain)

Scenario 4: $(Z_1, Z_2) = (-1.2, -0.9)$ - $W_1 = -1.2$, $W_2 = -0.6 - 0.779 = -1.379$ - $S_1(1) = 100 \exp[0.06 - 0.24] = 83.53$ - $S_2(1) = 150 \exp[0.06875 - 0.3448] = 117.92$ - $V_4 = 0.6(83.53) + 0.4(117.92) = \$97.29$ - Loss: $120 - 97.29 = \$22.71$

Scenario 5: $(Z_1, Z_2) = (0.2, 0.7)$ - $W_1 = 0.2$, $W_2 = 0.1 + 0.606 = 0.706$ - $S_1(1) = 100 \exp[0.06 + 0.04] = 110.52$ - $S_2(1) = 150 \exp[0.06875 + 0.1765] = 191.87$ - $V_5 = 0.6(110.52) + 0.4(191.87) = \$143.06$ - Loss: $-\$23.06$ (gain)

Sorted losses: {$-35.53, $-23.47, $-23.06, $-9.78, $22.71}

95% VaR: 95th percentile = 5th value = **$22.71**

Answer: The 1-year 95% VaR is **$22.71.** Note that 4 out of 5 scenarios resulted in gains, reflecting the positive expected returns. ∎

Example 5.8: Growth vs. Level Damage Functions

Problem: Explain why using a damage function that impacts the growth rate of GDP (like the BHM model) instead of the level would likely result in a higher Climate VaR over a long time horizon. Provide a numerical example with a 20-year horizon.

Solution:

Level effect (DICE-type):

$$GDP_t = GDP_0 \text{¿}$$

where d is the constant damage to the level.

Growth effect (BHM-type):

$$GDP_t = GDP_0 \text{¿}$$

where δ is the constant damage to the growth rate.

Numerical Example: - $GDP_0 = \$1000B$ - Baseline growth: $g = 0.03$ (3%) - Damage: $d = 0.05$ (5% level loss) or $\delta = 0.005$ (0.5% growth reduction) - Horizon: $T = 20$ years

Level effect:

$$GDP_{20}^{level} = 1000 \times \text{¿}$$

$$\text{¿} 1000 \times 1.806 \times 0.95 = \$1715.7B$$

$$\text{Loss} = 1000 \times 1.806 - 1715.7 = 1806 - 1715.7 = \$90.3B$$

Growth effect:

$$GDP_{20}^{growth} = 1000 \times \text{¿}$$

$$¿1000 \times 1.639 = \$1639\,B$$

$$\text{Loss} = 1806 - 1639 = \$167\,B$$

Ratio of losses:

$$\frac{167}{90.3} = 1.85$$

Answer: The growth effect produces **85% larger losses** after 20 years. This is because growth effects compound over time: each year's reduced growth affects the base for all subsequent years. For longer horizons (e.g., 50 years), the ratio would be even larger. This explains why Climate VaR estimates using BHM-type models are typically much higher than those using DICE-type models. ∎

5.6 Supplementary Problems

Basic Problems (1-5)

- For a Monte Carlo simulation with $N = 400$ scenarios and sample standard deviation $s = 80$, calculate the standard error of the mean estimate.

- How many scenarios are required to reduce the standard error to 1.0 if the population standard deviation is $\sigma = 50$?

- Given sorted losses {\$10, \$20, \$30, \$40, \$50, \$60, \$70, \$80, \$90, \$100}, calculate the 80% VaR and 80% ES.

- If the damage function coefficient doubles (from 0.01 to 0.02), by what factor does the VaR increase (assuming quadratic damage function)?

- Verify that for $Z \sim N(0,1)$, the antithetic pair $(Z, -Z)$ has correlation -1.

Intermediate Problems (6-10)

(f) A simulation uses 1000 scenarios and estimates VaR_{95} = \$500 with 95% confidence interval [\$450, \$550]. A colleague argues that 10,000 scenarios are needed for regulatory approval. Estimate the new confidence interval width.

(g) Implement the control variate method to estimate $E[e^{2Z}]$ where $Z \sim N(0,1)$, using $Y = Z$ as the control variate (with known $E[Z]=0$). Derive the optimal $\beta^{¿}$.

(h) For importance sampling from $g(x) = N(\mu_g, 1)$ to estimate tail probabilities under $f(x) = N(0,1)$, derive the optimal μ_g for estimating $P(X > c)$ where c is large.

(i) A portfolio has 3 assets with weights (0.5, 0.3, 0.2) and individual VaR_{95} values of (\$100, \$80, \$60). Assuming perfect positive correlation, what is the portfolio VaR? What if they are independent?

(j) Prove that Expected Shortfall is a coherent risk measure (satisfies monotonicity, sub-additivity, positive homogeneity, and translation invariance), while VaR is not (fails sub-additivity).

Advanced Problems (11-15)

(k) **Quasi-Monte Carlo:** Research and explain how low-discrepancy sequences (e.g., Sobol sequences) can achieve faster convergence than standard Monte Carlo. What is the theoretical convergence rate?

(l) **Nested simulation:** For calculating VaR of a portfolio containing options (which themselves require Monte Carlo pricing), develop a nested simulation algorithm and analyze its computational complexity.

(m) **Adaptive sampling:** Design an algorithm that dynamically allocates more samples to regions of the parameter space where the loss function has high variance or where we need more precision (e.g., near the VaR threshold).

(n) **Kernel density estimation:** Instead of using the empirical distribution, fit a kernel density estimator to the simulated loss distribution. Derive the formula for VaR and ES under the KDE, and discuss the bias-variance tradeoff in bandwidth selection.

(o) **Convergence diagnostics:** Develop a statistical test to determine whether N scenarios are sufficient. Consider using the Kolmogorov-Smirnov test to compare loss distributions from two independent simulation runs, or bootstrap methods to estimate the sampling distribution of VaR.

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Chapter 6: Partial Differential Equations in Climate Finance

6.1 The Black-Scholes-Merton Equation

Partial Differential Equations (PDEs) are a cornerstone of quantitative finance, primarily used for the pricing of derivative securities. The most famous of these is the Black-Scholes-Merton (BSM) equation, which provides a theoretical estimate of the price of European-style options.

The BSM equation is derived under a set of idealizing assumptions, including that the underlying asset price follows a Geometric Brownian Motion (GBM) with constant drift and volatility. The derivation relies on forming a risk-free portfolio by combining the derivative and the underlying asset, and arguing that, in the absence of arbitrage opportunities, this portfolio must earn the risk-free rate of return.

Definition 6.1 (The Black-Scholes-Merton Equation): For a derivative with price $V(S,t)$, where S is the price of the underlying asset and t is time, the BSM equation is:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \text{(Eq. 6.1)}$$

Where: - V is the price of the derivative - S is the price of the underlying asset - t is time - r is the risk-free interest rate - σ is the volatility of the underlying asset

6.2 A Climate-Adjusted Black-Scholes PDE

To incorporate climate risk into derivative pricing, we can adapt the BSM framework. We start by replacing the standard GBM with the climate-driven SDE introduced in Chapter 4:

$$dS_t=\mu(T_t)S_t\,dt+\sigma(T_t)S_t\,dW_t \text{(Eq. 6.2)}$$

Here, the drift (μ) and volatility (σ) are functions of a temperature process, T_t. For this derivation, we will assume T_t is a deterministic function of time, $T(t)$, based on a given climate scenario.

Theorem 6.1 (The Climate-Adjusted PDE for Derivative Pricing)

Statement: Let $V(S,t)$ be the price of a derivative on an underlying asset S, whose price follows the climate-driven SDE above. The price V must satisfy the following PDE:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma¿$$

Proof:

- Construct a portfolio, Π, consisting of one derivative, V, and a short position in Δ units of the underlying asset, S:

$$\Pi=V-\Delta S$$

- The change in the value of this portfolio, $d\Pi$, is given by:

$$d\Pi=dV-\Delta dS$$

- Using Itô's Lemma for $V(S,t)$, where S follows the climate-driven SDE (with drift $a=\mu(T(t))S$ and diffusion $b=\sigma(T(t))S$):

$$dV=¿$$

- Substitute dV and dS into the expression for $d\Pi$:

$$d\Pi=¿$$

$$-\Delta\{\mu(T(t))Sdt+\sigma(T(t))Sd\,W_t\}$$

- To make the portfolio risk-free, we must eliminate the stochastic term containing $d\,W_t$. This is achieved by setting:

$$\Delta=\frac{\partial V}{\partial S}$$

- With this choice of Δ, the portfolio becomes instantaneously risk-free, and its dynamics are purely deterministic:

$$d\Pi=¿$$

- In the absence of arbitrage, a risk-free portfolio must earn the risk-free interest rate, r. Therefore:

$$d\Pi = r\Pi dt = r(V - \Delta S)\,dt = r\left(V - S\frac{\partial V}{\partial S}\right)dt$$

- Equating the two expressions for $d\Pi$:

¿

- Rearranging the terms yields the Climate-Adjusted PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma ¿$$

- ∎

Note: The original drift $\mu(T(t))$ does not appear in the final equation, a key feature of risk-neutral pricing. However, the climate impact persists through the temperature-dependent volatility term, $\sigma(T(t))$.

6.3 Numerical Methods for Solving Climate-Finance PDEs

Because the coefficient $\sigma(T(t))$ is a function of time, the Climate-Adjusted PDE generally does not have a simple analytical solution like the standard BSM equation. Therefore, we must turn to numerical methods, such as Finite Difference Methods (FDM).

FDM involves discretizing the continuous PDE on a grid of points in the (S, t) plane. The partial derivatives are replaced with finite difference approximations.

6.3.1 Finite Difference Approximations

(a) Time derivative (forward difference):

$$\frac{\partial V}{\partial t} \approx \frac{V(i, j+1) - V(i, j)}{\Delta t} \text{(Eq. 6.4)}$$

(b) First space derivative (central difference):

$$\frac{\partial V}{\partial S} \approx \frac{V(i+1, j) - V(i-1, j)}{2\Delta S} \text{(Eq. 6.5)}$$

(c) Second space derivative:

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V(i+1,j) - 2V(i,j) + V(i-1,j)}{\text{¿¿}}$$

Substituting these approximations into the PDE allows one to solve for the derivative value V at each grid point, typically by working backward from the known terminal condition (e.g., the payoff of an option at expiration).

6.3.2 Explicit and Implicit Schemes

1. **Explicit FDM:** Solves for $V(i,j)$ directly in terms of values at the next time step $(j+1)$. It is easy to implement but is only stable under certain conditions on Δt and ΔS.

Stability condition (von Neumann):

$$\Delta t \leq \text{¿¿}$$

2. **Implicit FDM:** Leads to a system of linear equations that must be solved at each time step. It is more complex to implement but is unconditionally stable, making it more robust.

3. **Crank-Nicolson Method:** A weighted average of explicit and implicit schemes, offering second-order accuracy in both time and space:

$$\frac{V^{j+1} - V^{j}}{\Delta t} = \frac{1}{2}\left[L V^{j+1} + L V^{j}\right] \text{(Eq. 6.8)}$$

where L is the spatial differential operator.

6.4 Worked Examples

Example 6.1: Pricing a Climate-Sensitive Option

Problem: Set up the problem for pricing a European call option with strike $K = 100$ and maturity $T_{final} = 1$ year on an asset whose volatility increases with temperature according to $\sigma(T(t)) = 0.20 + 0.01\,T(t)$. The temperature path is $T(t) = 2t/T_{final}$. The risk-free rate is $r = 0.05$.

Solution:

(a) The PDE to Solve:

Substituting $T(t)=2t$:

$$\sigma(t)=0.20+0.01(2t)=0.20+0.02t$$

The PDE becomes:

$$\frac{\partial V}{\partial t}+\frac{1}{2}¿$$

(b) Boundary and Terminal Conditions:

11. **Terminal Condition (at** $t=1$**):** $V(S,1)=\max(S-100,0)$
12. **Boundary Condition 1 (at** $S=0$**):** $V(0,t)=0$
13. **Boundary Condition 2 (for** $S\to\infty$**):** $V(S,t)\to S-100e^{-0.05(1-t)}$

(c) Numerical Method Setup (Implicit FDM):

- Discretize the (S,t) domain into a grid with steps $\Delta S=5$ and $\Delta t=0.01$
- Replace the partial derivatives in the PDE with their implicit finite difference approximations
- This results in a system of linear equations at each time step j of the form:

$$A_j V_j = V_{j+1}+b_j$$

- where V_j is the vector of option values at time step j, A_j is a tridiagonal matrix whose coefficients depend on $\sigma(t_j)$, and b_j contains the boundary conditions.
- Starting with the known terminal condition $V_{T_{final}}$, solve this system of equations backward in time from $j=100$ down to $j=0$ to find the option price $V(S,0)$ today.

Answer: The setup is complete. Numerical solution would require implementation of the implicit FDM algorithm. ∎

Example 6.2: Explicit FDM Stability Analysis

Problem: For the standard Black-Scholes PDE with $\sigma = 0.25$, $r = 0.05$, determine the maximum time step Δt that ensures stability of the explicit FDM scheme if $\Delta S = 2$ and $S_{max} = 200$.

Solution:

The stability condition for explicit FDM is:

$$\Delta t \leq ¿¿$$

Substituting the values:

$$\Delta t \leq ¿¿$$

$$¿ \frac{4}{0.0625 \times 40000}$$

$$¿ \frac{4}{2500}$$

$$¿ 0.0016$$

Answer: The maximum time step for stability is $\Delta t = 0.0016$ years (approximately 0.58 days). This is very restrictive, which is why implicit methods are often preferred despite their computational complexity. ∎

Example 6.3: Crank-Nicolson Implementation

Problem: Write out the Crank-Nicolson scheme explicitly for the Black-Scholes PDE with constant coefficients. Show that it is second-order accurate in both time and space.

Solution:

The Black-Scholes PDE is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Let $LV = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV$.

The Crank-Nicolson scheme is:

$$\frac{V_i^{j+1} - V_i^j}{\Delta t} = \frac{1}{2}\left[LV_i^{j+1} + LV_i^j\right]$$

Expanding the spatial operator using finite differences:

$$LV_i^j = \frac{1}{2}\sigma^2 S_i^2 \frac{V_{i+1}^j - 2V_i^j + V_{i-1}^j}{\text{¿¿}}$$

The full scheme becomes:

$$V_i^{j+1} - V_i^j = \frac{\Delta t}{2}\left[LV_i^{j+1} + LV_i^j\right]$$

Rearranging:

$$V_i^{j+1} - \frac{\Delta t}{2} LV_i^{j+1} = V_i^j + \frac{\Delta t}{2} LV_i^j$$

This can be written in matrix form:

$$\left(I - \frac{\Delta t}{2} L\right) V^{j+1} = \left(I + \frac{\Delta t}{2} L\right) V^j$$

Accuracy: The Crank-Nicolson method is second-order accurate in time because it uses the average of the spatial operator at two time levels. Combined with second-order central differences in space, the overall scheme is $O\text{¿}$.

Answer: The Crank-Nicolson scheme is unconditionally stable and second-order accurate, making it the preferred method for many PDE applications in finance. ∎

Example 6.4: Climate-Dependent Dividends

Problem: Derive the Climate-Adjusted PDE for a derivative whose underlying asset is subject to both climate-dependent volatility $\sigma(T(t))$ and climate-dependent dividends $q(T(t))$.

Solution:

For an asset paying continuous dividends at rate q, the SDE is:

$$dS_t = (\mu(T(t)) - q(T(t)))S_t dt + \sigma(T(t))S_t dW_t$$

Following the same hedging argument as in Theorem 6.1:

- Construct portfolio: $\Pi = V - \Delta S$
- Apply Itô's Lemma to get dV
- Set $\Delta = \partial V / \partial S$ to eliminate stochastic term
- The risk-free portfolio must earn r, but now we must account for dividend income from the short position

The portfolio dynamics become:

$$d\Pi = dV - \Delta dS + q(T(t))\Delta S dt$$

The last term represents dividend income from the short position in the stock.

Following through the algebra:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma \text{¿}$$

Rearranging:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma \text{¿}$$

Answer: The Climate-Adjusted PDE with dividends is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma \text{¿}$$

The dividend yield $q(T(t))$ appears in the drift term, reducing the effective growth rate of the stock. ∎

Example 6.5: Climate-Dependent Risk-Free Rate

Problem: How would the PDE change if the risk-free rate, r, was also a function of temperature, $r(T(t))$? Provide economic intuition for why r might depend on climate.

Solution:

Modified PDE:

Following the same derivation, but now with $r = r(T(t))$:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma ¿$$

Economic Intuition:

The risk-free rate might depend on temperature for several reasons:

- **Central Bank Policy:** Central banks may adjust interest rates in response to climate-induced economic shocks (e.g., lowering rates after a climate disaster to stimulate recovery)
- **Inflation:** Climate change can affect inflation through:

  9. Food prices (agricultural productivity)
  10. Energy prices (transition costs)
  11. Supply chain disruptions

  Since $r_{nominal} = r_{real} + \pi$ (Fisher equation), climate-driven inflation changes affect r
- **Economic Growth:** If climate damages reduce GDP growth, the equilibrium real interest rate may decline (as predicted by growth models)
- **Risk Premium:** Sovereign risk premiums may increase for countries heavily exposed to climate risk, raising their “risk-free” rates

Functional Form Example:

$$r(T) = r_0 - \alpha T$$

where $\alpha > 0$ captures the negative impact of warming on the equilibrium interest rate.

Answer: The PDE becomes $\frac{\partial V}{\partial t}+\frac{1}{2}\sigma ¿$. Climate affects interest rates through central bank policy, inflation, growth, and risk premiums. ∎

Example 6.6: Rising Volatility Path and Option Prices

Problem: Explain intuitively why a rising volatility path ($d\sigma/dt>0$) due to climate change would lead to a higher price for a European call option compared to a constant volatility $\sigma=\sigma(T(0))$. Provide a numerical example.

Solution:

Intuition:

- **Convexity of Payoff:** The call option payoff $\max(S_T-K,0)$ is convex in S_T
- **Jensen's Inequality:** For a convex function f, $E[f(X)]\geq f(E[X])$. Higher volatility increases the spread of the distribution of S_T while keeping $E[S_T]$ constant (under risk-neutral measure)
- **Asymmetric Payoff:** The option benefits from upside moves but is protected from downside (payoff is zero, not negative). Higher volatility increases the probability of large upside moves, which increases option value
- **Time-Varying Volatility:** If volatility is rising over time, the later periods (closer to expiration) have higher volatility, which disproportionately affects the final distribution of S_T

Numerical Example:

Consider a call option with: - $S_0=100$, $K=100$, $T=1$ year, $r=0.05$

Case 1: Constant volatility - $\sigma(t)=0.20$ for all t - Black-Scholes price: $C=S_0 N(d_1)-K e^{-rT} N(d_2)$ - $d_1=\frac{\ln(S_0/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}=\frac{0+0.07}{0.20}=0.35$ - $d_2=d_1-\sigma\sqrt{T}=0.35-0.20=0.15$ - $C=100\times 0.6368-100 e^{-0.05}\times 0.5596=63.68-53.19=\$10.49$

Case 2: Rising volatility - $\sigma(t)=0.20+0.10t$ (rises from 20% to 30% over the year) - Average volatility: $\acute{\sigma}=0.25$ - Using average in Black-Scholes (approximation): - $d_1=\frac{0.07+0.03125}{0.25}=0.405$ - $d_2=0.405-0.25=0.155$ - $C\approx 100\times 0.6573-95.12\times 0.5616=65.73-53.42=\$12.31$

Answer: Rising volatility increases the call option price from \$10.49 to approximately \$12.31 (17% increase). This is because higher volatility later in the option's life increases the probability of finishing in-the-money, and the convex payoff structure means the option benefits more from increased upside than it loses from increased downside. ∎

6.5 Supplementary Problems

Basic Problems (1-5)

- Verify that the Black-Scholes PDE (Eq. 6.1) is satisfied by the European call option price formula $C=SN(d_1)-Ke^{-rT}N(d_2)$ by explicitly computing all partial derivatives.
- For the explicit FDM scheme, derive the update formula for V_i^j in terms of V_{i-1}^{j+1}, V_i^{j+1}, and V_{i+1}^{j+1}.
- Show that the Crank-Nicolson method reduces to the explicit method when the weighting parameter is 0 and to the implicit method when it is 1.
- For a European put option with payoff $\max(K-S_T,0)$, write down the terminal and boundary conditions for the Black-Scholes PDE.
- If volatility doubles from $\sigma=0.20$ to $\sigma=0.40$, by what factor does the stability condition $\Delta t\leq ¿$ change?

Intermediate Problems (6-10)

(f) Derive the Climate-Adjusted PDE for an American put option, which can be exercised at any time before expiration. How does the early exercise feature affect the PDE?

(g) Implement the explicit FDM scheme in pseudocode for pricing a European call option with constant volatility. Include the stability check.

(h) For the climate-dependent volatility $\sigma(T(t))=\sigma_0 e^{\alpha T(t)}$ where $T(t)=T_0+\beta t$, write out the full Climate-Adjusted PDE and discuss how the exponential volatility growth affects option prices.

(i) Prove that the Crank-Nicolson method is unconditionally stable using von Neumann stability analysis.

(j) For a barrier option that knocks out if S ever reaches a level B, modify the boundary conditions in the FDM scheme.

Advanced Problems (11-15)

(k) **Multi-dimensional PDE:** Derive the PDE for a derivative on two underlying assets, both subject to climate-dependent volatilities $\sigma_1(T(t))$ and $\sigma_2(T(t))$ with correlation $\rho(T(t))$.

(l) **American option pricing:** Develop a linear complementarity problem (LCP) formulation for American options under climate-dependent volatility, and describe how to solve it using the projected SOR method.

(m) **Stochastic volatility:** Extend the Climate-Adjusted PDE to the case where temperature itself follows a stochastic process $dT_t=\mu_T dt+\sigma_T dW_t^T$, resulting in a two-dimensional PDE in (S,T).

(n) **Jump-diffusion PDE:** Derive the PIDE (partial integro-differential equation) for option pricing when the underlying asset follows a jump-diffusion process with climate-dependent jump intensity $\lambda(T(t))$.

(o) **Convergence analysis:** Prove that the Crank-Nicolson method converges to the true solution of the Black-Scholes PDE with order O¿ as $\Delta t, \Delta S \to 0$.

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Chapter 7: Translating Climate Risk to Financial Statements

7.1 Mathematical Basis for Asset Impairment Testing (IAS 36)

International Accounting Standard 36 (IAS 36) requires an entity to test its assets for impairment. An asset is impaired when its carrying amount exceeds its recoverable amount. The recoverable amount is the higher of an asset's fair value less costs of disposal and its value in use.

Climate change can be a significant impairment indicator. For example, a factory located in a region with increasing flood risk may have its future cash-generating ability compromised. The mathematical link is established by modifying the calculation of the **value in use (VIU).**

Definition 7.1 (Value in Use): VIU is the present value of the future cash flows expected to be derived from an asset or cash-generating unit (CGU).

$$VIU = \sum_{t=1}^{T} \frac{E[CF_t]}{¿¿} ¿$$

Theorem 7.1 (Climate-Adjusted Impairment Test)

Statement: An asset is impaired due to climate risk if its carrying amount (CA) is greater than its climate-adjusted recoverable amount. The climate-adjusted VIU is calculated by incorporating a climate damage function $D(T_t)$ into the cash flow projections.

$$VIU_{climate} = \sum_{t=1}^{T} \frac{E[CF_t] \times (1 - D(T_t))}{¿¿} ¿$$

An impairment loss is recognized if:

$$CA > \max(VIU_{climate}, \text{Fair Value} - \text{Costs}) \text{(Eq. 7.3)}$$

Proof:

- IAS 36 requires that cash flow projections used to calculate VIU are based on "reasonable and supportable assumptions"

- Projections of future climate change and their physical impacts (e.g., from GCMs) and economic consequences (e.g., from damage functions) represent the best available evidence and thus form a reasonable and supportable basis for adjusting future cash flows

- As proven in Theorem 2.1, the climate-impacted cash flow at time t is:

$$CF_t' = CF_t \times (1 - D(T_t))$$

- Substituting CF_t' into the standard VIU formula yields the $VIU_{climate}$

- The standard impairment test ($CA >$¿ Recoverable Amount) is then applied using this climate-adjusted VIU ∎

7.2 Fair Value Adjustments (IFRS 13)

IFRS 13 defines fair value as the price that would be received to sell an asset or paid to transfer a liability in an orderly transaction between market participants at the measurement date. It establishes a fair value hierarchy:

14. **Level 1:** Quoted prices in active markets for identical assets
15. **Level 2:** Inputs other than quoted prices that are observable
16. **Level 3:** Unobservable inputs

Climate risk primarily affects Level 2 and Level 3 valuations, where models are used.

Mathematical Application: For Level 3 valuations, which rely on unobservable inputs, entities must develop their own models. The Climate-Adjusted DCF model (Theorem 2.1) is a primary tool for this.

$$\text{Fair Value (Level 3)} = \sum_{t=1}^{T} \frac{E[CF_t] \times (1 - D(T_t))}{¿¿} ¿$$

Here, the unobservable inputs include: - The choice of climate scenario and corresponding temperature path, $T(t)$ - The parameters of the damage function, $D(T)$ - The climate risk premium embedded in the discount rate, r_c

IFRS 13 requires disclosure of the sensitivity of Level 3 valuations to changes in these unobservable inputs, which directly corresponds to the sensitivity analysis discussed in Chapter 8.

7.3 Contingent Liabilities (IAS 37)

IAS 37 defines a contingent liability as a possible obligation that arises from past events and whose existence will be confirmed only by the occurrence or non-occurrence of one or more uncertain future events not wholly within the control of the entity.

Climate change can create contingent liabilities, such as the risk of future carbon taxes, fines for exceeding emissions limits, or climate-related litigation.

Definition 7.2 (Mathematical Provision for a Contingent Liability): A provision should be recognized if the probability of an outflow of resources is greater than 50%. The amount recognized should be the best estimate of the expenditure required to settle the obligation. Mathematically, this is the probability-weighted expected loss.

$$\text{Provision} = E[\text{Loss}] = P(\text{Event}) \times E[\text{Loss} \vee \text{Event}] \text{(Eq. 7.5)}$$

Where: - $P(\text{Event})$ is the probability of the climate-related event occurring (e.g., the probability of a carbon tax being enacted) - $E[\text{Loss} \vee \text{Event}]$ is the expected financial loss if the event occurs

7.4 Worked Examples

Example 7.1: Calculating an Asset Impairment Charge

Problem: A company owns a coastal asset with a carrying amount of $10M. Its projected cash flows are $1.2M per year in perpetuity, and the discount rate is 10%. Due to sea-level rise, there is a 20% probability that the asset will be permanently flooded and generate zero cash flow in 5 years. Calculate the impairment loss.

Solution:

(a) Calculate Baseline VIU (no climate risk):

$$VIU_{baseline} = \frac{\$1.2M}{0.10} = \$12M$$

The asset is not impaired at baseline.

(b) Calculate Climate-Adjusted VIU:

The cash flows can be modeled as an expectation: - Years 1-5: $CF = \$1.2M$ (guaranteed) - Year 6 onwards: $E[CF] = (0.8 \times \$1.2M) + (0.2 \times \$0) = \$0.96M$

$$VIU_{climate} = \sum_{t=1}^{5} \frac{\$1.2M}{¿¿} ¿$$

$$¿ \$4.549M + \frac{\$9.6M}{1.6105} = \$4.549M + \$5.96M = \$10.509M$$

(c) Impairment Test:

The recoverable amount is \$10.509M. The carrying amount is \$10M.

Since $CA < VIU_{climate}$ (\$10M < \$10.509M), there is **no impairment loss** to be recognized under this specific scenario. ∎

Example 7.2: Quantifying a Contingent Liability

Problem: A company emits 100,000 tonnes of CO_2 per year. There is a 60% probability that a carbon tax will be introduced in 3 years. If enacted, the tax is expected to be \$50 per tonne. The company's discount rate is 8%. What is the present value of the provision to be recognized?

Solution:

(a) Probability of Event: $P(\text{Event}) = 0.60$

Since this is > 50%, a provision must be considered.

(b) Expected Loss if Event Occurs:

Annual Loss = 100,000 tonnes × \$50/tonne = \$5M per year

(c) Present Value of the Liability at t=3:

Assuming the tax is perpetual, the value of the liability at the time of introduction ($t=3$) is:

$$V_3 = \frac{\$5M}{0.08} = \$62.5M$$

(d) Probability-Weighted Present Value Today (t=0):

$$\text{Provision} = P(\text{Event}) \times \frac{V_3}{\text{¿¿}}$$

$$\text{¿}0.60 \times \frac{\$62.5M}{\text{¿¿}}$$

$$\text{¿}0.60 \times \frac{\$62.5M}{1.2597} = 0.60 \times \$49.61M = \$29.77M$$

Answer: The provision to be recognized is **$29.77M.** This is the best estimate of the present value of the expenditure required to settle the future obligation. ∎

Example 7.3: Impairment Threshold Calculation

Problem: In Example 7.1, at what probability of flooding would the impairment loss be exactly zero (i.e., the carrying amount would equal the climate-adjusted VIU)?

Solution:

Let p be the probability of flooding.

The climate-adjusted VIU is:

$$VIU_{climate}(p) = \sum_{t=1}^{5} \frac{\$1.2M}{\text{¿¿}} \text{¿}$$

$$\text{¿}\$4.549M + \frac{(1-p) \times \$12M}{1.6105}$$

$$\text{¿}\$4.549M + \$7.451M \times (1-p)$$

$$\text{¿}\$4.549M + \$7.451M - \$7.451M \times p$$

$$\text{¿}\$12M - \$7.451M \times p$$

For no impairment, we need:

$$VIU_{climate}(p)=CA=\$10M$$

$$\$12M-\$7.451M\times p=\$10M$$

$$\$7.451M\times p=\$2M$$

$$p=\frac{\$2M}{\$7.451M}=0.2684=26.84\%$$

Answer: At a flooding probability of **26.84%**, the impairment loss would be exactly zero. For any probability above this threshold, an impairment loss must be recognized. ∎

Example 7.4: Fair Value Sensitivity Disclosure

Problem: A company has an asset with climate-adjusted fair value of \$4M, calculated using a damage function $D(T)=0.02T^2$ with expected temperature $T=3°C$. IFRS 13 requires sensitivity disclosure. Calculate the fair value if temperature is 2.5°C and 3.5°C, assuming baseline cash flows are \$600K per year in perpetuity with $r=0.12$.

Solution:

Baseline fair value (no climate):

$$FV_0=\frac{\$600K}{0.12}=\$5M$$

Fair value with climate adjustment:

$$FV(T)=FV_0\times(1-D(T))=\$5M\times(1-0.02T^2)$$

At T = 2.5°C:

$$D(2.5)=0.02\times¿$$

$$FV(2.5)=\$5M\times(1-0.125)=\$5M\times0.875=\$4.375M$$

At T = 3.0°C (base case):

$$D(3.0)=0.02\times9=0.18=18\%$$

$$FV(3.0) = \$5M \times 0.82 = \$4.1M$$

At T = 3.5°C:

$$D(3.5) = 0.02 \times 12.25 = 0.245 = 24.5\%$$

$$FV(3.5) = \$5M \times 0.755 = \$3.775M$$

Sensitivity table:

Temperature (°C)	Damage (%)	Fair Value ($M)	Change from Base
2.5	12.5%	4.375	+$0.275M (+6.7%)
3.0 (base)	18.0%	4.100	-
3.5	24.5%	3.775	-$0.325M (-7.9%)

Answer: The fair value ranges from **$3.775M to $4.375M** for a ±0.5°C temperature range, representing a ±7-8% sensitivity. This disclosure helps users understand the uncertainty in the Level 3 valuation. ∎

Example 7.5: Multi-Scenario Impairment Analysis

Problem: A company has an asset with carrying amount of $5M. Its climate-adjusted VIU is calculated under three NGFS scenarios:

17. **Net Zero 2050:** VIU = $5.5M (probability 30%)
18. **Delayed Transition:** VIU = $4.8M (probability 50%)
19. **Current Policies:** VIU = $3.5M (probability 20%)

The fair value less costs of disposal is $4.2M. What is the expected impairment loss?

Solution:

(a) Determine impairment under each scenario:

Scenario 1 (Net Zero): - Recoverable amount = max($5.5M, $4.2M) = $5.5M - Impairment = max($5M - $5.5M, 0) = $0

Scenario 2 (Delayed Transition): - Recoverable amount = max($4.8M, $4.2M) = $4.8M - Impairment = max($5M - $4.8M, 0) = $0.2M

Scenario 3 (Current Policies): - Recoverable amount = max($3.5M, $4.2M) = $4.2M - Impairment = max($5M - $4.2M, 0) = $0.8M

(b) Expected impairment:

$$E[\text{Impairment}] = 0.30 \times \$0 + 0.50 \times \$0.2M + 0.20 \times \$0.8M$$

$$¿0 + \$0.1M + \$0.16M = \$0.26M$$

(c) Accounting treatment:

IAS 36 requires impairment based on the most likely scenario or management’s best estimate, not the probability-weighted average. However, the expected value provides useful information for disclosure.

If management selects the “Delayed Transition” scenario as most likely (50% probability), the impairment would be **$0.2M.**

Answer: The expected impairment is **$0.26M**, but the recognized impairment would typically be **$0.2M** based on the most likely scenario. The range ($0 to $0.8M) should be disclosed to show climate scenario sensitivity. ∎

Example 7.6: Carbon Tax Provision with Uncertainty

Problem: A company emits 100,000 tonnes of CO_2 per year. There is a 60% probability that a carbon tax will be introduced in 3 years. If enacted, the tax could be $30/tonne (40% probability), $50/tonne (40% probability), or $80/tonne (20% probability). The company’s discount rate is 8%. Calculate the provision, accounting for both enactment uncertainty and tax rate uncertainty.

Solution:

(a) Expected tax rate (conditional on enactment):

$$E[\text{Tax} \vee \text{Enacted}] = 0.40 \times \$30 + 0.40 \times \$50 + 0.20 \times \$80$$

$$¿\,\$12 + \$20 + \$16 = \$48/\text{tonne}$$

(b) Expected annual cost (conditional on enactment):

$$\text{Annual Cost} = 100{,}000 \times \$48 = \$4.8\,M$$

(c) PV of perpetual liability at t=3:

$$V_3 = \frac{\$4.8\,M}{0.08} = \$60\,M$$

(d) Probability-weighted PV today:

$$\text{Provision} = P(\text{Enacted}) \times \frac{V_3}{¿¿}$$

$$¿\,0.60 \times \frac{\$60\,M}{1.2597} = 0.60 \times \$47.62\,M = \$28.57\,M$$

(e) Alternative calculation (full probability tree):

Scenario	Probability	Tax Rate	Annual Cost	PV at t=3	PV at t=0
No tax	40%	\$0	\$0	\$0	\$0
Tax @ \$30	24% (60%×40%)	\$30	\$3M	\$37.5M	\$8.94M
Tax @ \$50	24% (60%×40%)	\$50	\$5M	\$62.5M	\$14.90M
Tax @ \$80	12% (60%×20%)	\$80	\$8M	\$100M	\$11.91M

$$\text{Total Expected PV} = \$0 + \$8.94\,M + \$14.90\,M + \$11.91\,M = \$35.75\,M$$

Note: The discrepancy ($28.57M vs. $35.75M) arises because the first method uses the expected tax rate, while the second properly accounts for the non-linearity (perpetuity formula is non-linear in the tax rate).

Answer: The correct provision is **$35.75M,** calculated using the full probability tree to properly account for the non-linear relationship between tax rates and liability values. ∎

Example 7.7: Goodwill Impairment from Climate Risk

Problem: A company acquired a business for $50M, of which $15M was allocated to goodwill. The cash-generating unit (CGU) to which the goodwill belongs has identifiable net assets of $35M. Due to emerging climate regulations, the CGU's projected cash flows have declined. The climate-adjusted VIU is now $42M, and fair value less costs is $40M. Calculate the goodwill impairment.

Solution:

(a) Carrying amount of CGU:

$$CA_{CGU} = \text{Identifiable net assets} + \text{Goodwill} = \$35M + \$15M = \$50M$$

(b) Recoverable amount:

$$\text{Recoverable Amount} = \max(VIU, FV - Costs) = \max(\$42M, \$40M) = \$42M$$

(c) Total impairment:

$$\text{Total Impairment} = CA_{CGU} - \text{Recoverable Amount} = \$50M - \$42M = \$8M$$

(d) Allocation of impairment:

IAS 36 requires impairment to be allocated: 1. First, to goodwill 2. Then, to other assets pro rata based on carrying amounts

Goodwill impairment: min($15M, $8M) = **$8M**

Since the total impairment ($8M) is less than the goodwill ($15M), the entire impairment is allocated to goodwill.

New carrying amounts: - Goodwill: \$15M - \$8M = \$7M - Identifiable net assets: \$35M (unchanged) - Total CGU: \$42M

Answer: The goodwill impairment is **\$8M.** This is a permanent write-down; IAS 36 prohibits reversal of goodwill impairments in subsequent periods, even if climate conditions improve. ∎

7.5 Supplementary Problems

Basic Problems (1-5)

- A company has an asset with carrying value of \$5M. Its climate-adjusted VIU is calculated to be \$4M, and its fair value less costs of disposal is \$4.2M. What is the impairment loss?
- For a contingent liability with 45% probability of occurrence and expected loss of \$10M if it occurs, should a provision be recognized under IAS 37? Explain.
- Calculate the baseline VIU for an asset generating \$800K per year for 10 years with a discount rate of 9%.
- If a carbon tax of \$40/tonne is enacted with certainty in 2 years, and a company emits 50,000 tonnes/year, what is the PV of the perpetual liability at $r = 7\%$?
- An asset has carrying amount \$8M and climate-adjusted VIU of \$7.5M. If the damage function coefficient increases by 20%, reducing VIU to \$7M, what is the additional impairment?

Intermediate Problems (6-10)

(f) Derive the formula for the impairment threshold probability in Example 7.3 for the general case where the asset generates cash flow CF for n years before potential failure, with discount rate r and baseline perpetuity value V_0.

(g) A company must choose between two climate scenarios for impairment testing: RCP4.5 (VIU = \$6M) and RCP8.5 (VIU = \$4.5M). If the carrying amount is \$5M, under which scenario(s) is impairment required? How should management choose?

(h) For the carbon tax provision in Example 7.6, calculate the 95% confidence interval for the provision amount, assuming the tax rate distribution is approximately normal with mean \$48 and standard deviation \$18.

(i) A CGU has goodwill of \$20M and identifiable assets of \$60M. Climate-adjusted VIU is \$70M. If regulations change and VIU drops to \$65M next year, what is the goodwill impairment in each year?

(j) Develop a Monte Carlo algorithm to estimate the distribution of impairment losses for an asset subject to uncertain climate damages, discount rates, and carrying amounts.

Advanced Problems (11-15)

(k) **IFRS 13 Level 3 valuation:** Derive the sensitivity of fair value to the damage function parameter β in $D(T) = \beta T^2$ for a perpetual cash flow stream. Show that $\frac{\partial FV}{\partial \beta} = \frac{-CF \cdot T^2}{r}$.

(l) **Contingent liability with compounding:** Modify Example 7.6 to account for annual emissions growth of 3% and a tax that escalates at 2% per year after enactment. Derive the provision formula.

(m) **Portfolio-level impairment:** A company has 10 assets, each with independent flood risk. Develop a model to calculate the expected total impairment and its variance, accounting for correlation in climate damages.

(n) **Reversal of impairment:** IAS 36 allows reversal of impairment (except goodwill) if conditions improve. Derive the conditions under which a previously impaired asset should have its impairment reversed, accounting for climate scenario updates.

(o) **Deferred tax implications:** When an impairment loss is recognized for accounting purposes but not tax purposes (creating a deductible temporary difference), a deferred tax asset arises. Develop a framework for calculating the net impairment impact including deferred tax effects under IAS 12.

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Chapter 8: Uncertainty Propagation and Sensitivity Analysis

8.1 Mathematical Theory of Error Propagation

Climate-financial models are composed of a long chain of uncertain parameters, from physical climate sensitivity to economic damage coefficients. Understanding how uncertainty in these inputs propagates to the output (e.g., asset valuation or Climate VaR) is critical for robust decision-making. The mathematical theory of error propagation provides a framework for this analysis.

Theorem 8.1 (General Formula for Variance of a Function)

Statement: Let Y be a function of multiple uncertain variables, $Y=f(X_1, X_2, \ldots, X_n)$. If the variables X_i have variances σ_i^2 and covariances $Cov(X_i, X_j)$, then the variance of Y, σ_Y^2, can be approximated by a first-order Taylor series expansion:

$$\sigma_Y^2 \approx \sum_{i=1}^{n} \left[\left(\frac{\partial f}{\partial X_i} \right)^2 \sigma_i^2 \right] + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \left[\frac{\partial f}{\partial X_i} \frac{\partial f}{\partial X_j} Cov(X_i, X_j) \right] \text{(Eq. 8.1)}$$

Proof Outline:

- Expand the function f around the mean values of X_i, μ_i:

$$f(X) \approx f(\mu) + \sum_{i=1}^{n} \frac{\partial f}{\partial X_i} (X_i - \mu_i)$$

- The variance of Y is E¿. Substituting the Taylor expansion and taking the expectation yields the formula above. The partial derivatives are evaluated at the mean values of the input variables. ∎

If the input variables are uncorrelated, the covariance terms are zero, and the formula simplifies to:

$$\sigma_Y^2 \approx \sum_{i=1}^{n} \left[\left(\frac{\partial f}{\partial X_i} \right)^2 \sigma_i^2 \right] \text{(Eq. 8.2)}$$

This shows that the output variance is a weighted sum of the input variances, where the weights are the squares of the partial derivatives (sensitivities) of the function with respect to each input.

8.2 Sensitivity Analysis (Greeks)

In finance, the sensitivity of a derivative's price to a change in an input parameter is known as a "Greek." We can adapt this concept to climate-financial models to understand which parameters are the most significant drivers of risk.

Definition 8.1 (Climate-Financial Greeks): Let V be the value of a climate-sensitive asset or portfolio. We can define the following sensitivities:

(a) **Delta** (δ): Sensitivity to a change in the underlying asset price (already standard).

$$\delta = \frac{\partial V}{\partial S} \text{(Eq. 8.3)}$$

(b) **Temperature Theta** (Θ_T): Sensitivity to a change in the current temperature anomaly.

$$\Theta_T = \frac{\partial V}{\partial T} \text{(Eq. 8.4)}$$

(c) **Lambda's Lambda** (Λ_λ): Sensitivity to a change in the climate feedback parameter (λ).

$$\Lambda_\lambda = \frac{\partial V}{\partial \lambda} \text{(Eq. 8.5)}$$

(d) **Forcing Vega** (ν_F): Sensitivity to a change in the volatility (standard deviation) of the radiative forcing estimate.

$$\nu_F = \frac{\partial V}{\partial \sigma_F} \text{(Eq. 8.6)}$$

These sensitivities are the partial derivatives that appear in the error propagation formula and are crucial for identifying the largest sources of uncertainty in a model.

8.3 Sobol Indices for Global Sensitivity Analysis

For non-linear models, first-order sensitivity (partial derivatives) may not capture the full picture. Sobol indices provide a variance-based global sensitivity measure.

Definition 8.2 (Sobol Indices): For a function $Y = f(X_1, \ldots, X_n)$, the first-order Sobol index for variable X_i is:

$$S_i = \frac{Var[E[Y \vee X_i]]}{Var[Y]} \text{(Eq. 8.7)}$$

This represents the fraction of output variance explained by X_i alone.

The total-order Sobol index is:

$$S_T^i = 1 - \frac{Var[E[Y \vee X_{\sim i}]]}{Var[Y]} \text{(Eq. 8.8)}$$

where $X_{\sim i}$ denotes all variables except X_i. This captures the total contribution of X_i, including interactions.

8.4 Confidence Intervals for Financial Risk Estimates

As the output of a Monte Carlo simulation is itself a random sample, the resulting risk estimates (like VaR and ES) are subject to estimation error. We can construct confidence intervals for these estimates.

Theorem 8.2 (Confidence Interval for Value-at-Risk)

Statement: For a VaR estimate at confidence level α obtained from N simulations, the standard error of the VaR estimate, $SE(VaR)$, is given by:

$$SE(VaR_\alpha) = \frac{1}{f(VaR_\alpha)} \sqrt{\frac{\alpha(1-\alpha)}{N}} \text{(Eq. 8.9)}$$

Where $f(x)$ is the probability density function of the loss distribution at the VaR point.

An approximate $(1-\beta)$ confidence interval for the true VaR is then:

$$VaR_\alpha \pm Z_{1-\beta/2} \times SE(VaR_\alpha) \text{(Eq. 8.10)}$$

Proof Outline: This result is derived from order statistics. The uncertainty in the VaR estimate depends on how many data points fall around the α-th quantile. The density $f(VaR_\alpha)$ reflects this: a lower density (flatter tail) means more uncertainty in the location of the quantile, leading to a larger standard error.

Since the true density $f(x)$ is often unknown, it can be estimated from the simulation results using kernel density estimation or by assuming a parametric distribution (e.g., normal) for the losses. ∎

8.5 Worked Examples

Example 8.1: Error Propagation for a Simple Climate-Damage Model

Problem: An asset's value is modeled as $V=100\times(1-0.01T^2)$. The temperature anomaly T is uncertain, with a mean of 3°C and a standard deviation of 0.5°C. Estimate the standard deviation of the asset's value.

Solution:

(a) Calculate the sensitivity ($\partial V/\partial T$):

$$\frac{\partial V}{\partial T}=\frac{d}{dT}[100-T^2]=-2T$$

(b) Evaluate the sensitivity at the mean temperature:

At $T=3$:

$$\frac{\partial V}{\partial T}=-2\times 3=-6$$

(c) Apply the error propagation formula (for one variable):

$$\sigma_V^2\approx\left(\frac{\partial V}{\partial T}\right)^2\sigma_T^2$$

¿¿

(d) Calculate the standard deviation of V:

$$\sigma_V=\sqrt{9}=\$3$$

Answer: The 0.5°C uncertainty in temperature translates to an approximate **$3 uncertainty** in the asset's value. ∎

Example 8.2: Confidence Interval for a VaR Estimate

Problem: A Monte Carlo simulation with $N = 10{,}000$ runs yields a 99% VaR of \$500M. The loss distribution is assumed to be normal with a standard deviation of \$150M. Calculate the 95% confidence interval for this VaR estimate.

Solution:

(a) Estimate the density $f(VaR)$:

For a normal distribution $N(\mu, \sigma^2)$, the PDF is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp ¿$$

We need to estimate μ. From $VaR = \mu + Z_\alpha \sigma$, we have:

$$\mu = VaR - Z_\alpha \sigma = 500 - 2.33 \times 150 = 150.5\,M$$

Now, evaluate $f(500)$:

$$f(500) = \frac{1}{150\sqrt{2\pi}} \exp ¿$$

(b) Calculate the Standard Error of VaR:

$$SE(VaR_{99\%}) = \frac{1}{0.00039}\sqrt{\frac{0.99 \times 0.01}{10000}}$$

$$\approx 2564 \times 0.000995 \approx \$2.55\,M$$

(c) Calculate the 95% Confidence Interval:

The Z-score for a 95% CI is $Z_{0.975} = 1.96$.

$$CI = \$500\,M \pm 1.96 \times \$2.55\,M = \$500\,M \pm \$5.0\,M$$

$$CI = [\$495\,M, \$505\,M]$$

Answer: We are 95% confident that the true 99% VaR lies between **\$495M and \$505M.** ∎

Example 8.3: Multi-Variable Error Propagation

Problem: Consider the equilibrium temperature model $\Delta T = F/\lambda$. Assume F and λ are uncorrelated. F has a mean of 4 W/m² and a standard deviation of 0.5 W/m². λ has a mean of 1.0 W/m²/K and a standard deviation of 0.2 W/m²/K. Calculate the approximate variance of ΔT.

Solution:

(a) Calculate partial derivatives:

$$\frac{\partial(\Delta T)}{\partial F} = \frac{\partial}{\partial F}\left(\frac{F}{\lambda}\right) = \frac{1}{\lambda}$$

$$\frac{\partial(\Delta T)}{\partial \lambda} = \frac{\partial}{\partial \lambda}\left(\frac{F}{\lambda}\right) = \frac{-F}{\lambda^2}$$

(b) Evaluate at mean values:

At $\acute{F}=4$, $\acute{\lambda}=1.0$:

$$\frac{\partial(\Delta T)}{\partial F} ¿_{mean} = \frac{1}{1.0} = 1.0$$

$$\frac{\partial(\Delta T)}{\partial \lambda} ¿_{mean} = \frac{-4}{¿¿}$$

(c) Apply error propagation formula (uncorrelated variables):

$$\sigma_{\Delta T}^2 = \left(\frac{\partial(\Delta T)}{\partial F}\right)^2 \sigma_F^2 + \left(\frac{\partial(\Delta T)}{\partial \lambda}\right)^2 \sigma_\lambda^2$$

$$¿¿$$

$$¿1.0 \times 0.25 + 16.0 \times 0.04$$

$$¿0.25 + 0.64 = 0.89$$

(d) Standard deviation:

$$\sigma_{\Delta T} = \sqrt{0.89} = 0.943\,°C$$

(e) Mean temperature:

$$\acute{\Delta T}=\frac{\acute{F}}{\acute{\lambda}}=\frac{4}{1.0}=4.0\,°C$$

Answer: The equilibrium temperature is $4.0\pm 0.94\,°C$ (mean ± 1 std dev). Note that uncertainty in λ contributes more to output variance (0.64) than uncertainty in F (0.25), despite λ having smaller absolute uncertainty. This is because ΔT is more sensitive to λ (partial derivative of -4.0 vs. 1.0). ∎

Example 8.4: Sobol Indices Calculation

Problem: For the damage function $D(T,\beta)=\beta T^2$ where $T\sim N(3,0.5^2)$ and $\beta\sim N(0.02,0.005^2)$ (independent), calculate the first-order Sobol indices for T and β.

Solution:

(a) Calculate total variance:

Using error propagation:

$$Var[D]=\left(\frac{\partial D}{\partial T}\right)^2 Var[T]+\left(\frac{\partial D}{\partial \beta}\right)^2 Var[\beta]$$

$$\frac{\partial D}{\partial T}=2\beta T\,,\frac{\partial D}{\partial \beta}=T^2$$

At means: $\acute{T}=3$, $\acute{\beta}=0.02$:

$$\frac{\partial D}{\partial T}¿_{mean}=2\times 0.02\times 3=0.12$$

$$\frac{\partial D}{\partial \beta}¿_{mean}=3^2=9$$

$$Var[D]=¿$$

$$¿\,0.0144\times 0.25+81\times 0.000025$$

$$¿\,0.0036+0.002025=0.005625$$

(b) Calculate conditional variances:

$$Var[E[D \vee T]] = Var[\beta T^2] = T^4 Var[\beta]$$

At mean T:

$$Var[E[D \vee T]] = 3^4 \times 0.005^2 = 81 \times 0.000025 = 0.002025$$

$$Var[E[D \vee \beta]] = Var[2\beta T \times T] = \beta^2 \times 4 Var[T]$$

At mean β:

$$Var[E[D \vee \beta]] = 0.02^2 \times 4 \times 0.5^2 = 0.0004 \times 1 = 0.0004$$

Wait, this approach is incorrect. Let me use the proper formula:

$$Var[E[D \vee T]] = E_T[Var_\beta[D \vee T]] = E_T[T^4 Var[\beta]] = E[T^4] \times Var[\beta]$$

For $T \sim N(3, 0.5^2)$:

$$E[T^4] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4 = 3^4 + 6 \times 9 \times 0.25 + 3 \times 0.0625 = 81 + 13.5 + 0.1875 = 94.6875$$

Actually, this is getting complex. Let me use simulation approach in practice, but for pedagogical purposes:

(c) Sobol indices (approximate using partial derivatives):

$$S_T \approx ¿¿$$

$$S_\beta \approx ¿¿$$

Answer: Temperature T explains approximately **64%** of the variance in damages, while the damage coefficient β explains **36%.** This indicates that reducing uncertainty in temperature projections would have a larger impact on reducing damage uncertainty than refining the damage function parameter. ∎

Example 8.5: Monte Carlo Standard Error Reduction

Problem: If you increase the number of Monte Carlo simulations (N) from 10,000 to 40,000, by what factor would you expect the standard error of the VaR estimate to decrease? Verify with the formula.

Solution:

From Theorem 8.2, the standard error is:

$$SE(VaR) = \frac{1}{f(VaR)} \sqrt{\frac{\alpha(1-\alpha)}{N}}$$

The standard error is proportional to $1/\sqrt{N}$.

(a) Ratio of sample sizes:

$$\frac{N_{new}}{N_{old}} = \frac{40{,}000}{10{,}000} = 4$$

(b) Ratio of standard errors:

$$\frac{SE_{new}}{SE_{old}} = \sqrt{\frac{N_{old}}{N_{new}}} = \sqrt{\frac{10{,}000}{40{,}000}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Answer: The standard error would decrease by a factor of **2** (i.e., it would be halved). This is the $\sqrt{N}$ rule: to reduce standard error by half, you must quadruple the number of simulations. ∎

Example 8.6: Sensitivity Analysis for Agricultural Portfolio

Problem: Why is a sensitivity analysis with respect to the damage function parameters (e.g., the β coefficients in the BHM model) crucial for understanding the risk of a portfolio heavily invested in agriculture? Provide a quantitative example.

Solution:

Conceptual Answer:

Agricultural assets are highly sensitive to temperature changes due to: 1. **Non-linear yield response:** Crops have optimal temperature ranges; deviations reduce yields non-linearly 2. **Regional heterogeneity:** Different crops and regions have different β coefficients 3. **Parameter uncertainty:** BHM coefficients have wide confidence intervals 4. **Compounding effects:** Yield impacts compound over time through growth effects

Quantitative Example:

Consider a farmland portfolio with annual revenue $R=\$10M$ and damage function:

$$D(T)=\beta_1 T+\beta_2 T^2$$

Base case: $\beta_1=-0.04$, $\beta_2=0.002$, $T=2°C$ above optimal

$$D(2)=-0.04\times 2+0.002\times 4=-0.08+0.008=-0.072=-7.2\%$$

$$\text{Revenue}=\$10M\times(1-0.072)=\$9.28M$$

Sensitivity to β_2 (quadratic term):

If $\beta_2=0.003$ (50% higher):

$$D(2)=-0.08+0.012=-0.068=-6.8\%$$

$$\text{Revenue}=\$9.32M$$

$$\text{Change}=+\$40K$$

If $\beta_2=0.001$ (50% lower):

$$D(2)=-0.08+0.004=-0.076=-7.6\%$$

$$\text{Revenue}=\$9.24M$$

$$\text{Change}=-\$40K$$

For 20-year NPV at r = 8%:

The sensitivity is magnified:

$$\text{NPV sensitivity} = \$\,40\,K \times 1 - ¿¿$$

Answer: A 50% uncertainty in the quadratic damage coefficient translates to approximately **±$393K** uncertainty in portfolio NPV. For large agricultural portfolios, this uncertainty can be in the tens of millions. Therefore, sensitivity analysis is crucial for: - Identifying which parameters most affect valuation - Prioritizing research to reduce parameter uncertainty - Setting appropriate risk reserves - Informing hedging strategies ■

Example 8.7: Confidence Interval Width Comparison

Problem: Compare the 95% confidence interval widths for VaR estimates at 95% and 99% confidence levels, assuming the same loss distribution and sample size. Which VaR estimate has more sampling uncertainty?

Solution:

From Theorem 8.2:

$$SE(VaR_\alpha) = \frac{1}{f(VaR_\alpha)}\sqrt{\frac{\alpha(1-\alpha)}{N}}$$

For VaR₉₅:

$$SE(VaR_{0.95}) = \frac{1}{f(VaR_{0.95})}\sqrt{\frac{0.95 \times 0.05}{N}} = \frac{1}{f(VaR_{0.95})}\sqrt{\frac{0.0475}{N}}$$

For VaR₉₉:

$$SE(VaR_{0.99}) = \frac{1}{f(VaR_{0.99})}\sqrt{\frac{0.99 \times 0.01}{N}} = \frac{1}{f(VaR_{0.99})}\sqrt{\frac{0.0099}{N}}$$

Ratio of numerators:

$$\frac{\sqrt{0.0475}}{\sqrt{0.0099}} = \frac{0.218}{0.0995} = 2.19$$

However, the density f also matters. For a normal distribution, the tail is thinner (lower density) at higher quantiles.

Numerical example: Assume $N(0,1)$ loss distribution, $N=10{,}000$:

20. VaR_{95} = 1.645, $f(1.645)=0.103$
21. VaR_{99} = 2.326, $f(2.326)=0.027$

$$SE(VaR_{0.95})=\frac{1}{0.103}\times 0.0218=0.212$$

$$SE(VaR_{0.99})=\frac{1}{0.027}\times 0.00995=0.368$$

95% CI widths: - VaR_{95}: $2\times 1.96\times 0.212=0.831$ - VaR_{99}: $2\times 1.96\times 0.368=1.443$

Answer: The 99% VaR estimate has **74% wider confidence interval** than the 95% VaR estimate. This is because: 1. The $\alpha(1-\alpha)$ term is smaller for 99% (more extreme quantile) 2. The density f is much lower in the tail (less data near the quantile) 3. The second effect dominates, making extreme quantiles harder to estimate precisely ■

8.6 Supplementary Problems

Basic Problems (1-5)

- For the function $Y=aX+b$ where a and b are constants and X has variance σ_X^2, derive the variance of Y using the error propagation formula.

- Calculate the Temperature Theta (Θ_T) for an asset valued at $V=100(1-0.015T^2)$ when $T=2.5\,°C$.

- If a VaR estimate has standard error of $5M, what is the 90% confidence interval? (Use $Z_{0.95}=1.645$)

- For two uncorrelated variables X_1 and X_2 with equal variances σ^2, and a function $Y=X_1+X_2$, show that $\sigma_Y=\sqrt{2}\sigma$.

- Explain intuitively why the standard error of VaR decreases as $1/\sqrt{N}$ rather than $1/N$.

Intermediate Problems (6-10)

(f) Derive the error propagation formula for the product of two uncertain variables: $Z=XY$. Show that if X and Y are uncorrelated:

$$\frac{\sigma_Z^2}{Z^2}=\frac{\sigma_X^2}{X^2}+\frac{\sigma_Y^2}{Y^2}$$

(g) For the climate-adjusted DCF formula $V=\sum_{t=1}^{T}\frac{CF_t(1-D(T_t))}{¿¿}¿$, derive expressions for the sensitivities $\partial V/\partial r$ and $\partial V/\partial T_t$.

(h) Calculate the first-order Sobol index for the forcing parameter F in the equilibrium temperature model $\Delta T=F/\lambda$, given $F\sim N(4,0.5^2)$ and $\lambda\sim N(1,0.2^2)$ (independent).

(i) A simulation with $N=5000$ yields VaR$_{95}$ = \$200M with 95% CI of [\$190M, \$210M]. How many simulations are needed to reduce the CI width to ±\$5M?

(j) Prove that for a linear function $Y=\sum_{i=1}^{n}a_i X_i$, the first-order Taylor approximation in the error propagation formula is exact (no approximation error).

Advanced Problems (11-15)

(k) **Second-order error propagation:** Derive the second-order Taylor expansion for $Var[f(X)]$ and show that it includes terms involving $E¿$ (skewness) and $E¿$ (kurtosis).

(l) **Correlated inputs:** For the equilibrium temperature model $\Delta T=F/\lambda$, assume F and λ have correlation $\rho=-0.3$ (negative because higher forcing often comes with higher uncertainty in feedback). Recalculate the variance of ΔT including the covariance term.

(m) **Bootstrap confidence intervals:** Describe a bootstrap procedure to estimate the confidence interval for VaR without assuming a parametric form for the loss distribution. Implement the algorithm in pseudocode.

(n) **Total Sobol indices:** For a function $Y=f(X_1,X_2,X_3)$ with interactions, explain why $\sum_{i=1}^{3}S_i$ (sum of first-order indices) may be less than 1, and how total indices S_T^i account for this.

(o) **Optimal allocation of simulation budget:** You have a computational budget for $N=10{,}000$ simulations to estimate both VaR_{95} and VaR_{99}. Derive the optimal allocation N_1 and N_2 (where $N_1+N_2=10{,}000$) to minimize the sum of squared standard errors, accounting for the different densities at each quantile.

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Chapter 9: Advanced Problems and Case Studies

This chapter applies the mathematical frameworks developed in previous chapters to complex, integrated problems. These case studies require combining concepts from climate science, stochastic modeling, financial theory, and accounting to address realistic challenges in quantifying climate risk. Each problem is designed to test deep understanding and require rigorous mathematical reasoning.

Part I: Asset Valuation Under Climate Risk

Case Study 1: Valuing a Coastal Real Estate Portfolio with Sea-Level Rise

Problem Statement: A real estate investment trust (REIT) holds a portfolio of 50 coastal properties valued at $500M. The properties are at various elevations. Sea-level rise (SLR) projections show a mean rise of 0.5m by 2050 with a standard deviation of 0.15m. Properties below critical thresholds will be abandoned. Model the portfolio value accounting for this risk.

Mathematical Formulation:

22. **Sea-Level Rise Process:** Model SLR as a stochastic process with drift:
23. $dH(t) = \mu_H \, dt + \sigma_H \, dW_t$

    Where $\mu_H = 0.01$m/year, $\sigma_H = 0.003$m/year.

24. **Property-Specific Inundation: Let h_i be the elevation of property i above current sea level. Property i is inundated when $H(t) \geq h_i$.**

25. **Valuation PDE: For each property, the value $V_i(H, t)$ satisfies:**

26. $\partial V_i/\partial t + (1/2)\sigma_{H^2}(\partial^2 V_i/\partial H^2) + \mu_H(\partial V_i/\partial H) - rV_i + CF_i = 0$

    With boundary condition: $V_i(h_i, t) = 0$ (worthless when inundated).

27. **Portfolio Aggregation:**
28. $V_{portfolio} = \sum(i=1 \text{ to } 50) \; V_i(H_0, 0)$

Required Analysis: - Solve the PDE numerically using finite difference methods - Calculate the expected loss: E[Loss] = $V_{baseline}$ - $V_{climate}$ - Compute the 95% VaR for portfolio value at t=30 years - Perform sensitivity analysis on μ_H and σ_H

Extension: Incorporate the option to invest in flood defenses at cost $C_{defense}$ that raises the inundation threshold by Δh. Determine the optimal defense investment strategy.

Case Study 2: Agricultural Asset Valuation with Temperature Volatility

Problem Statement: A farmland investment fund owns 10,000 hectares producing corn. Historical yield is 10 tonnes/hectare with revenue $200/tonne. The Burke-Hsiang-Miguel model predicts yield changes based on temperature. Current average temperature is 18°C. Model the farm value over 20 years under RCP4.5 (moderate emissions).

Mathematical Model:

29. **Yield Function:**
30. $Y(T) = Y_{baseline} * (1 + \beta_1(T - T_{baseline}) + \beta_2(T - T_{baseline})^2)$

    With $\beta_1 = 0.04$, $\beta_2 = -0.002$ (calibrated for corn).

31. **Temperature Path:**
32. $T(t) = T_0 + \alpha t + \sigma_T\sqrt{t} * Z$

    Where $\alpha = 0.05$°C/year (RCP4.5 trend), $\sigma_T = 0.3$°C, $Z \sim N(0,1)$.

33. **Cash Flow:**
34. CF(t) = Area * Y(T(t)) * Price - Costs
35. **Farm Value:**
36. $V_0 = \sum(t=1 \text{ to } 20)\ E[CF(t)] / (1+r)^t$

Required Analysis: - Calculate expected value using Monte Carlo simulation (10,000 paths) - Determine the probability that farm value falls below $15M - Calculate the temperature sensitivity: dV/dT at T = 20°C - Compare results with a linear damage function

Extension: Add a jump-diffusion component for extreme heat events with intensity $\lambda = 0.1$/year and yield loss of 30% per event.

Case Study 3: Energy Company Valuation Under Carbon Pricing

Problem Statement: A coal-fired power plant generates 1,000 MW with capacity factor 0.75. It emits 0.9 tCO_2/MWh. Current electricity price is $50/MWh, operating cost $25/MWh. The plant has 15 years remaining life. Model its value under three NGFS scenarios: Net Zero 2050, Delayed Transition, and Current Policies.

Mathematical Framework:

37. **Carbon Price Trajectories:**
    1. Net Zero 2050: $P_{carbon}(t) = 75 * e^{(0.08t)}$ $/$tCO_2$
    2. Delayed Transition: $P_{carbon}(t) = 0$ for t<5, then $150 * e^{(0.12(t-5))}$
    3. Current Policies: $P_{carbon}(t) = 20 * (1 + 0.02t)$
38. **Operating Profit:**
39. $\pi(t) = (P_{elec} - C_{op} - E_{intensity} * P_{carbon}(t)) * \text{Generation}$

    Where Generation = Capacity * CF * 8760 hours/year.
40. **Stranding Condition:** Plant is stranded (shut down) when $\pi(t) < 0$.
41. **Plant Value:**
42. $V = \sum(t=1 \text{ to } T_{strand}) \max(\pi(t), 0) / (1+r)^t - \text{Decommissioning}_{cost}$

Required Analysis: - Calculate plant value under each scenario - Determine the stranding date for each scenario - Calculate the stranded asset loss: $V_{baseline} - V_{scenario}$ - Perform sensitivity analysis on electricity price and carbon price growth rates

Extension: Model the option to retrofit with carbon capture (CCS) at cost $500M, reducing emissions by 90%. Determine if the real option value justifies investment under each scenario.

Case Study 4: Insurance Company Portfolio Under Physical Risk

Problem Statement: An insurer has a $10B property portfolio with 60% in coastal regions. Historical annual loss ratio is 5%. Climate models project a 50% increase in hurricane intensity and 20% increase in frequency by 2050. Model the impact on loss ratios and required capital.

Mathematical Model:

43. **Loss Distribution (Current):** Annual losses L ~ Compound Poisson with:

    1. Frequency: $\lambda_0 = 2$ events/year
    2. Severity: S ~ Pareto(α=2.5, x_m=$100M)

44. **Climate-Adjusted Parameters:**

45. $\lambda(t) = \lambda_0 * (1 + 0.2 * t/30)$ $S(t) \sim \text{Pareto}(\alpha=2.5, x_m(t) = x_m * (1 + 0.5 * t/30))$
46. **Total Annual Loss:**
47. $L_{total}(t) = \sum(i=1 \text{ to } N(t)) \, S_i(t)$

    Where $N(t) \sim \text{Poisson}(\lambda(t))$.

48. **Required Capital (99.5% VaR):**
49. $\text{Capital}(t) = \text{VaR}_{99}.5\%(L_{total}(t))$

Required Analysis: - Simulate loss distributions for years 2025, 2035, 2045, 2055 - Calculate required capital increase over time - Determine the premium increase needed to maintain 15% ROE - Calculate the probability of ruin (losses > capital) over 30 years

Extension: Model the correlation between hurricane losses and equity market returns. Recalculate required capital accounting for this correlation.

Case Study 5: Renewable Energy Project Valuation with Weather Uncertainty

Problem Statement: A solar farm project requires $100M investment and will generate electricity for 25 years. Expected capacity factor is 25% with 1,500 MW capacity. Electricity price is $60/MWh. Climate change may reduce solar irradiance by 5% (±3%) due to increased cloud cover. Value the project.

Mathematical Framework:

50. **Generation Model:**
51. $E(t) = Capacity * CF(t) * 8760$

    Where $CF(t) = CF_0 * (1 + \Delta CF)$, $\Delta CF \sim N(-0.05, 0.03^2)$.

52. **Revenue:**
53. $R(t) = E(t) * P_{elec}(t)$

    With $P_{elec}(t) = P_0 * (1 + g)^t$, $g = 0.02$.

54. **Project NPV:**
55. $NPV = -I_0 + \sum(t=1 \text{ to } 25)\ (R(t) - O\&M(t)) / (1+WACC)^t$
56. **Climate Risk Adjustment:** Use Monte Carlo to simulate ΔCF and calculate E[NPV] and Std[NPV].

Required Analysis: - Calculate base-case NPV (no climate impact) - Calculate expected NPV with climate uncertainty - Determine the probability that NPV < 0 - Calculate the climate risk premium: $WACC_{climate} - WACC_{base}$

Extension: Add a battery storage option (cost $20M, capacity 500 MWh) that increases revenue by 15%. Determine if the real option value justifies the investment.

Part II: Financial Institution Risk Management

Case Study 6: Bank Loan Portfolio Stress Testing

Problem Statement: A bank has a $50B corporate loan portfolio with the following sector exposure: - Oil & Gas: 20% ($10B) - Manufacturing: 30% ($15B) - Real Estate: 25% ($12.5B) - Services: 25% ($12.5B)

Stress test under NGFS “Delayed Transition” scenario with sudden $200/t$CO_2$ carbon tax.

Mathematical Framework:

57. **Sector-Specific Emissions Intensity:**

1. Oil & Gas: 500 tCO_2/\$M revenue
2. Manufacturing: 150 tCO_2/\$M revenue
3. Real Estate: 50 tCO_2/\$M revenue
4. Services: 20 tCO_2/\$M revenue

58. **Cost Shock:**

59. $\Delta Cost_i = Emissions_{intensity\_i} * P_{carbon} * Revenue_i$

60. **EBIT Impact:**

61. $EBIT_{new} = EBIT_{baseline} - \Delta Cost$

62. **PD Model (Merton):**

63. $PD = \Phi(-DD)$ $DD = (\ln(V/D) + (\mu - 0.5\sigma^2)T) / (\sigma\sqrt{T})$

    Where DD is distance to default, updated with new EBIT.

64. **Expected Loss:**

65. $EL = \sum(i=1 \text{ to } N)\ PD_i * LGD_i * EAD_i$

Required Analysis: - Calculate baseline PD for each sector (assume $DD_{baseline} = 3.0$) - Calculate stressed PD after carbon tax - Calculate incremental expected loss: $\Delta EL = EL_{stressed} - EL_{baseline}$ - Determine required additional loan loss provisions

Extension: Model second-order effects where manufacturing firms pass through 50% of carbon costs to customers, reducing demand by elasticity $\epsilon = -0.8$.

Case Study 7: Pension Fund Asset Allocation Under Climate Risk

Problem Statement: A \$20B pension fund has 60% equities, 30% bonds, 10% alternatives. The fund must meet \$800M annual liabilities for 30 years. Incorporate climate risk into asset allocation using mean-variance optimization.

Mathematical Framework:

66. **Asset Returns (Climate-Adjusted):**

67. $R_{equities}(T) = \mu_{eq} + \beta_{eq} * D(T) + \sigma_{eq} * \epsilon_{eq}$ $R_{bonds} = r_f + \sigma_{bonds} * \epsilon_{bonds}$ $R_{alternatives}(T) = \mu_{alt} + \beta_{alt} * D(T) + \sigma_{alt} * \epsilon_{alt}$

    Where D(T) is the BHM damage function.

68. **Temperature Scenarios:**

    1. Optimistic: T = 1.5°C, p = 0.2
    2. Base: T = 2.5°C, p = 0.5
    3. Pessimistic: T = 4.0°C, p = 0.3

69. **Portfolio Optimization:**

70. min $\sigma_{p^2} = w'\Sigma w$ subject to: $E[R_p] \geq R_{target}$ $\sum w_i = 1$, $w_i \geq 0$

    Where Σ includes climate-induced correlations.

71. **Funding Ratio:**

72. FR(t) = Assets(t) / PV(Liabilities(t))

Required Analysis: - Calculate optimal weights under baseline (no climate risk) - Calculate optimal weights with climate risk - Simulate funding ratio paths (1,000 scenarios, 30 years) - Calculate probability of underfunding (FR < 0.8) at t=30

Extension: Add climate-linked bonds (green bonds) as a fourth asset class with return correlation -0.3 to temperature. Recalculate optimal allocation.

Case Study 8: Sovereign Debt Sustainability Under Climate Stress

Problem Statement: A small island nation has debt/GDP ratio of 80%. GDP is $10B, growing at 3%/year. Climate change threatens tourism (40% of GDP) and agriculture (15% of GDP). Model debt sustainability under RCP8.5.

Mathematical Framework:

73. **GDP Impact:**

74. $GDP(t) = GDP_0 * (1+g)^t * (1 - D_{tourism}(T(t)) * 0.4 - D_{agriculture}(T(t)) * 0.15)$

Where:

1. $D_{tourism}(T) = 0.05 * T^2$ (tourism highly temperature-sensitive)
2. $D_{agriculture}(T) = 0.02 * T^2$ (agriculture moderately sensitive)

75. **Temperature Path (RCP8.5):**
76. $T(t) = 1.2 + 0.08t + 0.5\sqrt{t} * Z, Z \sim N(0,1)$
77. **Debt Dynamics:**
78. $D(t+1) = D(t) * (1 + r) - PrimarySurplus(t)$ $PrimarySurplus(t) = \tau * GDP(t) - G(t)$

    Where $\tau = 0.25$ (tax rate), $G(t) = 0.20 * GDP(t)$ (government spending).

79. **Sustainability Condition:** Debt is sustainable if Debt/GDP ratio remains < 100%.

Required Analysis: - Simulate debt/GDP ratio for 30 years (1,000 Monte Carlo paths) - Calculate probability of debt crisis (Debt/GDP > 100%) - Determine the required fiscal adjustment (increase in τ or decrease in G) to maintain sustainability - Calculate the present value of expected climate damages

Extension: Add the option for the country to issue catastrophe bonds to finance climate adaptation (cost $500M, reduces $D_{tourism}$ and $D_{agriculture}$ by 30%). Determine if this is financially optimal.

Case Study 9: Credit Rating Migration Under Transition Risk

Problem Statement: A credit rating agency must update ratings for a portfolio of 100 corporate bonds across various sectors. Model rating migration probabilities under NGFS "Net Zero 2050" scenario.

Mathematical Framework:

80. **Rating Migration Matrix (Baseline): Standard transition matrix $P_{baseline}$ (8x8 for AAA to D).**

81. **Carbon Intensity Adjustment: For firm i with carbon intensity E_i (tCO_2/\$M revenue):**

82. $Adjustment_i = -\alpha * E_i * P_{carbon}(t)$

Where $\alpha = 0.0001$ (calibrated parameter).

83. **Adjusted Credit Spread:**
84. $Spread_{new} = Spread_{baseline} * \exp(Adjustment_i)$
85. **New Rating:** Map spread to rating using empirical spread-rating relationship.

86. **Portfolio Impact:**

87. $\Delta Value = \sum(i=1 \text{ to } 100)\ BondValue_i * (Spread_{new} - Spread_{baseline}) * Duration_i$

Required Analysis: - Calculate rating migration for each bond - Determine the percentage of bonds downgraded by 1, 2, 3+ notches - Calculate total portfolio value loss - Identify the sectors most affected

Extension: Model the feedback effect where rating downgrades increase borrowing costs, further impairing credit quality. Iterate until convergence.

Case Study 10: Equity Portfolio Climate Beta Estimation

Problem Statement: An equity portfolio manager holds 50 stocks across sectors. Estimate each stock's "climate beta" (sensitivity to climate risk factors) and construct a climate-hedged portfolio.

Mathematical Framework:

88. **Factor Model:**
89. $R_i = \alpha_i + \beta_{market} * R_{market} + \beta_{climate} * F_{climate} + \epsilon_i$

    Where $F_{climate}$ is a climate risk factor (e.g., carbon price changes).

90. **Climate Beta Estimation:** Use regression on historical data:
91. $\beta_{climate,i} = Cov(R_i, F_{climate}) / Var(F_{climate})$
92. **Climate-Hedged Portfolio:** Construct portfolio with zero climate beta:
93. $\sum(i=1 \text{ to } N)\ w_i * \beta_{climate,i} = 0$ subject to: $\sum w_i = 1$
94. **Tracking Error:**
95. $TE = \sqrt{(Var(R_{portfolio} - R_{benchmark}))}$

Required Analysis: - Estimate climate betas for all 50 stocks - Identify stocks with highest positive and negative climate betas - Construct minimum-variance climate-neutral portfolio - Calculate the tracking error vs. market-cap weighted benchmark

Extension: Add a constraint that the portfolio must achieve at least 90% of benchmark return while being climate-neutral. Solve the constrained optimization problem.

Part III: Corporate Financial Planning

Case Study 11: Capital Budgeting with Climate-Adjusted WACC

Problem Statement: A manufacturing firm is evaluating two projects: - Project A: Expand existing coal-based production ($200M investment, 10-year life) - Project B: Build new renewable-powered facility ($300M investment, 15-year life)

Determine which project to pursue, accounting for climate risk in the discount rate.

Mathematical Framework:

96. **Standard WACC:**
97. $WACC = w_e * r_e + w_d * r_d * (1-T)$
98. **Climate Risk Premium:**
99. $r_{climate} = \beta_{climate} * \lambda_{climate}$

    Where $\lambda_{climate}$ is the market price of climate risk (estimated at 2% for high-carbon projects).

100. **Climate-Adjusted WACC:**
101. $WACC_{climate} = WACC + r_{climate}$

    1. Project A: $\beta_{climate} = 1.5$ (high carbon intensity)
    2. Project B: $\beta_{climate} = -0.2$ (renewable, benefits from transition)

102. **Project NPV:**
103. $NPV = -I_0 + \sum(t=1 \text{ to } T)\ CF_t / (1 + WACC_{climate})$^t

Required Analysis: - Calculate NPV for both projects using standard WACC (assume 8%) - Calculate NPV using climate-adjusted WACC - Determine which project is preferred under each approach - Calculate the break-even climate risk premium where project choice switches

Extension: Model the uncertainty in future carbon prices as a geometric Brownian motion and calculate the option value of delaying the decision by 2 years.

Case Study 12: Supply Chain Climate Risk Assessment

Problem Statement: An automotive manufacturer sources components from 200 suppliers across 30 countries. 40% of suppliers are in regions with high physical climate risk (water stress, heat extremes). Model supply chain disruption risk.

Mathematical Framework:

104. **Supplier Disruption Probability:**
105. $P_{disruption,i} = P_{baseline} * (1 + \alpha * PhysicalRisk_i)$

    Where $PhysicalRisk_i$ is a composite index (0-1 scale).

106. **Production Impact: If supplier i is disrupted, production loss = $Criticality_i$ * ProductionValue.**
107. **Total Expected Loss:**
108. $E[Loss] = \sum(i=1 \text{ to } 200)\ P_{disruption,i} * Criticality_i * ProductionValue$
109. **Value-at-Risk:** Use Monte Carlo to simulate disruption scenarios and calculate 95% VaR.

Required Analysis: - Calculate expected annual loss from supply chain climate risk - Identify the top 10 suppliers contributing most to VaR - Model the benefit of diversifying suppliers (reducing concentration) - Calculate the optimal investment in supplier climate resilience

Extension: Model cascading failures where disruption of one supplier increases probability of disruption for dependent suppliers. Use network analysis to identify critical nodes.

Case Study 13: Real Estate Development Under Uncertain Regulation

Problem Statement: A developer is considering building a mixed-use development in a coastal area ($500M investment, 5-year construction, 30-year operating life). There is uncertainty about future building codes requiring flood protection.

Mathematical Framework:

110. **Regulatory Scenarios:**

    1. No new regulation: $p = 0.3$
    2. Moderate regulation (require 1m protection): $p = 0.5$, cost = $50M
    3. Strict regulation (require 2m protection): $p = 0.2$, cost = $120M

111. **Decision Tree:**

    1. Build now without protection
    2. Build now with 1m protection
    3. Build now with 2m protection
    4. Wait 2 years for regulatory clarity (opportunity cost = $30M)

112. **Project Value:**

113. $V = \sum(t=1 \text{ to } 30)\ NOI(t) / (1+r)^\wedge t - I_0 - Protection_{cost}$

    Where NOI = $40M/year.

114. **Expected Value:**

115. $E[V] = \sum p_{scenario} * V_{scenario}$

Required Analysis: - Calculate expected value for each decision - Determine optimal strategy - Calculate the value of waiting (option value) - Perform sensitivity analysis on regulation probabilities

Extension: Model the case where regulation is announced gradually (Bayesian updating). Use dynamic programming to find the optimal stopping time for the investment decision.

Case Study 14: Mining Company Closure Liability Valuation

Problem Statement: A mining company has a site that will be depleted in 15 years. Closure and remediation costs are estimated at $200M in today's dollars. Climate change may increase remediation costs due to extreme weather and water management challenges. Value the closure liability.

Mathematical Framework:

116. **Baseline Closure Cost:**
117. C_0 = $200M
118. **Climate Escalation:**
119. $C(T) = C_0 * (1 + \gamma * T)$

    Where $\gamma = 0.15$ (15% increase per °C of warming).

120. **Temperature at Closure (t=15):**
121. $T(15) \sim N(\mu_T, \sigma_T^2)$

    With $\mu_T = 2.0$°C, $\sigma_T = 0.5$°C.

122. **Present Value of Liability:**
123. PV = E[C(T(15))] / (1+r)^15
124. **Provision (IAS 37):** Company must recognize provision = PV today.

Required Analysis: - Calculate expected closure cost at t=15 - Calculate present value of liability - Determine the additional provision needed vs. baseline ($200M) - Calculate 90% confidence interval for the liability

Extension: Model the option to accelerate closure to t=10 (avoiding 5 years of climate escalation) at an additional cost of $30M. Determine if early closure is optimal.

Case Study 15: Utility Company Generation Mix Optimization

Problem Statement: An electric utility must plan its generation mix for 2030-2050. Current mix: 50% coal, 30% gas, 20% renewables. Model optimal transition path under carbon price uncertainty.

Mathematical Framework:

125. **Generation Technologies:** | Technology | CAPEX (¿kW¿∨OPEX¿/MWh) | Emissions (tCO_2/MWh) | Lifetime (years) | |————————|————————|————————|————————————-|————————————| | Coal | 2,000 | 25 | 0.9 | 40 | | Gas | 1,000 | 35 | 0.4 | 30 | | Solar | 1,200 | 0 | 0 | 25 | | Wind | 1,500 | 0 | 0 | 25 | | Battery | 1,000/kWh | 5 | 0 | 15 |

126. **Carbon Price Scenarios:**

    1. Low: P(t) = 50 * (1.05)^t
    2. Medium: P(t) = 100 * (1.08)^t
    3. High: P(t) = 200 * (1.12)^t

127. **Optimization Problem:**

128. min ∑(t=2030 to 2050) [CAPEX(t) + OPEX(t) + CarbonCost(t)] / (1+r)^t subject to: - Capacity ≥ Demand(t) + $\text{Reserve}_{\text{margin}}$ - $\text{Renewable}_{\text{share}}$ ≥ Target(t) - Reliability constraints

129. **Stochastic Programming:** Use scenario tree with branching carbon prices.

Required Analysis: - Solve deterministic optimization for medium carbon price scenario - Solve stochastic optimization with all three scenarios (equal probability) - Calculate the value of the stochastic solution (VSS) - Determine optimal retirement schedule for coal plants

Extension: Add the constraint that the utility must achieve net-zero emissions by 2050. Determine the least-cost pathway and the incremental cost vs. unconstrained solution.

Part IV: Advanced Quantitative Techniques

Case Study 16: Climate Tipping Point Modeling with Regime-Switching

Problem Statement: Model the risk of Atlantic Meridional Overturning Circulation (AMOC) collapse, which would cause severe economic disruption in Europe. Use a regime-switching model to capture the discontinuity.

Mathematical Framework:

130. **Two-Regime Model:**

    1. Regime 1 (Normal): GDP growth = 2.5%
    2. Regime 2 (Post-collapse): GDP growth = -1.0%

131. **Transition Probability:**

132. $P(\text{Switch} \mid T) = 1 / (1 + \exp(-\alpha(T - T_{threshold})))$

    Where $T_{threshold} = 3.5°C$, $\alpha = 2$.

133. **State Dynamics:**

134. $S(t+1) = S(t)$ if no switch $S(t+1) = 2$ if switch occurs

135. **GDP Process:**

136. $GDP(t+1) = GDP(t) * (1 + g_S(t+1) + \sigma * \varepsilon_t)$

137. **Asset Value:** European equity index value depends on GDP:

138. $V(t) = k * GDP(t)$

Required Analysis: - Simulate 10,000 paths of GDP and asset value over 50 years - Calculate the probability of regime switch by 2050, 2070, 2100 - Calculate expected asset value loss conditional on switch occurring - Determine the "tipping point VaR": the loss at 95% confidence

Extension: Model multiple tipping points (AMOC, Amazon rainforest, West Antarctic ice sheet) with correlation structure. Calculate joint probability of multiple tipping points.

Case Study 17: Optimal Carbon Tax Using DICE Model

Problem Statement: Using the DICE model framework, derive the optimal carbon tax trajectory that maximizes discounted global welfare. Compare with the social cost of carbon (SCC).

Mathematical Framework:

139. **Welfare Function:**

140. $W = \sum(t=1 \text{ to } T) [U(C(t)) * L(t)] / (1+\rho)^t$

    Where $U(C) = C^{(1-\eta)}/(1-\eta)$ (CRRA utility), $L(t)$ is population.

141. **Production:**

142. $Q(t) = A(t) * K(t)^\alpha * L(t)^{(1-\alpha)} * \Omega(T(t))$

Where $\Omega(T) = 1/(1 + a_2*T^2)$ is the damage function.

143. **Capital Accumulation:**

144. $K(t+1) = (1-\delta)*K(t) + I(t)$

145. **Consumption-Investment:**

146. $C(t) + I(t) + Abatement(t) = Q(t)$

147. **Climate Module:**

148. $T(t+1) = T(t) + \lambda^{(-1)} * (F(t) - \lambda T(t))$ $F(t) = F_{2x} \log_2(M(t)/M_{pre}\text{-industrial})$ $M(t+1) = M(t) + E(t) - Decay(M(t))$

149. **Optimization:**

150. max W subject to all constraints Control variables: I(t), Abatement(t)

151. **Optimal Carbon Tax:**

152. $\tau^*(t) = -\partial W/\partial E(t) = SCC(t)$

Required Analysis: - Solve the optimization problem numerically (use Lagrangian method) - Derive the optimal carbon tax path for 2025-2100 - Calculate the SCC for 2025, 2050, 2100 - Perform sensitivity analysis on discount rate (ρ = 0.5%, 1.5%, 3.0%)

Extension: Add uncertainty in climate sensitivity (λ) and damage function (a_2). Solve the stochastic optimization problem and compare optimal tax under uncertainty vs. certainty.

Case Study 18: Catastrophe Bond Pricing for Climate Risk

Problem Statement: An insurance company wants to issue a catastrophe bond to transfer hurricane risk. The bond pays 8% coupon but principal is forgiven if hurricane losses exceed $5B in a year. Price the bond accounting for climate change impacts on hurricane intensity.

Mathematical Framework:

153. **Hurricane Loss Model:** Annual losses L ~ Compound Poisson:

1. Frequency: $N \sim \text{Poisson}(\lambda(t))$
2. Severity: $X_i \sim \text{Pareto}(\alpha, x_m(t))$

154. **Climate Adjustment:**

155. $\lambda(t) = \lambda_0 * (1 + 0.02t)$ (frequency increase) $x_m(t) = x_{m,0} * (1 + 0.03t)$ (severity increase)
156. **Trigger Probability:**
157. $P(L > \$5B \mid t) = P(\sum(i=1 \text{ to } N(t)) X_i > \$5B)$
158. **Bond Cash Flows:**
159. CF_t = Coupon if $L \leq \$5B$ CF_t = Coupon + Principal if $L \leq \$5B$ and t = Maturity $CF_t = 0$ if $L > \$5B$ (principal forgiven)
160. **Bond Price:**
161. $P = \sum(t=1 \text{ to } T) E[CF_t] / (1+r_f + \text{spread})^t$

Required Analysis: - Calculate trigger probability for each year (t=1 to 5, 5-year bond) - Determine the fair spread above risk-free rate - Calculate expected loss to bondholders - Compare with traditional reinsurance pricing

Extension: Design a parametric trigger based on wind speed rather than actual losses. Determine the optimal trigger level that minimizes basis risk while maintaining attractive pricing.

Case Study 19: Green Bond Premium Estimation

Problem Statement: A corporation can issue either conventional bonds or green bonds (proceeds used for renewable energy projects). Estimate the “greenium” (green bond premium) and determine optimal issuance strategy.

Mathematical Framework:

162. **Bond Pricing:**
163. $P_{conventional} = \sum(t=1 \text{ to } T)\ C / (1+r_{conventional})^t + \text{Face} / (1+r_{conventional})^T$ $P_{green} = \sum(t=1 \text{ to } T)\ C / (1+r_{green})^t + \text{Face} / (1+r_{green})^T$
164. **Greenium:**
165. $\text{Greenium} = r_{conventional} - r_{green}$

166. **Investor Demand:** Model two investor types:

    1. ESG investors: Willing to accept lower yield (utility from green investment)
    2. Conventional investors: Yield-focused only

167. $Demand_{ESG}(r) = D_0 * exp(-β_{ESG} * (r - r_{min}))$ $Demand_{conventional}(r) = D_0 * exp(-β_{conv} * (r - r_{min}))$

168. **Market Clearing:**

169. $Supply = Demand_{ESG}(r_{green}) + Demand_{conventional}(r_{green})$

170. **Issuer Optimization:**

171. $max\ [P_{green} - P_{conventional}] - Certification_{cost}$

Required Analysis: - Estimate greenium from market data (assume 15-25 bps) - Calculate break-even certification cost - Determine optimal issuance size for green bonds - Model the impact of increasing ESG investor base on greenium

Extension: Add reputational risk where issuing green bonds commits the firm to emissions reduction targets. Model the trade-off between lower funding cost and future carbon price exposure.

Case Study 20: Climate Stress Testing with Macro-Financial Linkages

Problem Statement: A central bank conducts a climate stress test on the banking system. Model the transmission from climate scenarios to bank capital ratios through multiple channels.

Mathematical Framework:

172. **Climate Scenarios:** Use NGFS scenarios (Net Zero 2050, Delayed Transition, Current Policies).

173. **Macro Model:**

174. $GDP(t) = GDP(t-1) * (1 + g_{baseline} + Climate_{impact}(t) + Transition_{impact}(t))$ $Unemployment(t) = f(GDP_{growth}(t))$ $Interest_{rates}(t) = Taylor_{rule}(Inflation(t), Output_{gap}(t))$

175. **Climate Impact:**

176. $Climate_{impact}(t) = -β_{physical} * PhysicalDamage(T(t))$ $Transition_{impact}(t) = -β_{transition} * Carbon_{price}(t) * Carbon_{intensity\_economy}$

177. **Bank Balance Sheet:**

1. Assets: Loans to various sectors
2. Liabilities: Deposits, wholesale funding
3. Capital: Equity

178. **Credit Risk:**

179. $PD_{sector}(t) = PD_{baseline} * \exp(\beta_{sector} * GDP_{growth}(t) + \gamma_{sector} * Carbon_{price}(t))$

180. **Bank Losses:**

181. $Losses(t) = \sum_{sectors} [EAD_{sector} * PD_{sector}(t) * LGD_{sector}]$

182. **Capital Ratio:**

183. $CAR(t) = (Capital(t-1) - Losses(t) + Earnings(t)) / RWA(t)$

Required Analysis: - Simulate macro variables under each NGFS scenario - Calculate sector-specific PDs under each scenario - Determine bank losses and capital ratios over 30 years - Identify which banks fail (CAR < 8%) under each scenario

Extension: Model second-round effects where bank failures reduce credit supply, further depressing GDP. Iterate until convergence to capture amplification effects.

Case Study 21: Biodiversity Loss and Agricultural Finance

Problem Statement: An agricultural lender has $5B in loans to farms that depend on pollination services. Climate change threatens bee populations, reducing crop yields. Model the credit risk.

Mathematical Framework:

184. **Pollination Service Model:**

185. $Pollination_{effectiveness}(T) = P_0 * \exp(-\alpha * (T - T_{optimal})^2)$

Where $T_{optimal} = 15°C$, $\alpha = 0.05$.

186. **Crop Yield:**

187. $Yield(T) = Yield_{baseline} * Pollination_{effectiveness}(T) * Other_{factors}(T)$

188. **Farm Revenue:**

189. $Revenue(t) = Yield(T(t)) * Price(t) * Area$

190. **Debt Service Coverage Ratio (DSCR):**

191. $DSCR(t) = (Revenue(t) - Operating_{costs}) / Debt_{service}$

192. **Default Probability:**

193. $PD(t) = \Phi(-\log(DSCR(t)) / \sigma)$

Where Φ is the standard normal CDF.

194. **Portfolio Expected Loss:**

195. $EL = \sum(i=1 \text{ to } N)\ PD_i(t) * LGD * EAD_i$

Required Analysis: - Simulate temperature paths (RCP4.5 and RCP8.5) - Calculate pollination effectiveness over time - Determine farm-level PDs for t=10, 20, 30 years - Calculate portfolio expected loss and 95% unexpected loss

Extension: Model the option for farms to invest in alternative pollination methods (managed bee colonies) at cost $50K/farm. Determine optimal adoption rate from lender's perspective.

Case Study 22: Stranded Assets in Automotive Sector

Problem Statement: An auto manufacturer has $10B in assets dedicated to internal combustion engine (ICE) production. Model the stranding risk under accelerated electric vehicle (EV) adoption driven by climate policy.

Mathematical Framework:

196. **EV Adoption Curve (Bass Diffusion Model):**

197. $dN/dt = (p + q\textit{N/M})\ \ (M - N)$

Where:

1. N = cumulative EV adopters
2. M = market potential
3. p = innovation coefficient
4. q = imitation coefficient

198. **Climate Policy Impact:** Carbon price accelerates adoption:

199. $q(P_{carbon}) = q_{baseline} * (1 + \beta * P_{carbon})$

200. **ICE Asset Utilization:**

201. Utilization(t) = 1 - N(t)/M

202. **Asset Value:**

203. $V(t) = \sum(s=t \text{ to } T)\ CF(s) * Utilization(s) / (1+r)^\wedge(s-t)$

204. **Stranded Asset Loss:**

205. $Loss = V_{baseline}(0) - V_{climate}(0)$

Required Analysis: - Calibrate Bass model parameters (p=0.01, q=0.3 for EVs) - Simulate EV adoption under three carbon price scenarios - Calculate asset stranding dates (when utilization < 50%) - Determine present value of stranded asset losses

Extension: Model the manufacturer's option to repurpose ICE assets for EV production at conversion cost $2B. Use real options analysis to determine optimal conversion timing.

Case Study 23: Water Risk in Semiconductor Manufacturing

Problem Statement: A semiconductor fab requires 10 million gallons of water per day. It's located in a region where climate change is increasing water stress. Model the operational and financial risk.

Mathematical Framework:

206. **Water Availability Model:**

207. $W(t) = W_{baseline} * (1 + \alpha * P(t) - \beta * T(t))$

Where:

1. P(t) = precipitation anomaly
2. T(t) = temperature anomaly
3. $\alpha = 0.5$ (precipitation sensitivity)
4. $\beta = 0.1$ (temperature sensitivity)

208. **Water Stress Events:** Water stress occurs when W(t) < Requirement.

209. $P(\text{Stress} \mid t) = P(W(t) < 10\text{M gallons})$

210. **Production Impact:** During water stress, production reduced by:

211. $\text{Production}_{loss} = \min(1, (\text{Requirement} - W(t)) / \text{Requirement})$

212. **Revenue Impact:**

213. $\text{Revenue}_{loss}(t) = \text{Production}_{loss}(t) * \text{Revenue}_{per\_day} * \text{Days}_{stressed}$

214. **Mitigation Options:**

    1. Build water recycling facility: \$500M, reduces requirement by 40%
    2. Secure alternative water source: \$200M, provides 5M gallons/day backup

215. **NPV of Mitigation:**

216. $\text{NPV}_{mitigation} = -\text{Investment} + \sum(t=1 \text{ to } 20) \text{ Avoided}_{losses}(t) / (1+r)^t$

Required Analysis: - Simulate water availability under RCP4.5 (1,000 scenarios, 20 years) - Calculate expected annual revenue loss - Calculate 95% VaR for revenue loss - Determine optimal mitigation strategy (NPV-maximizing)

Extension: Model the correlation between water stress at this fab and other fabs in the region. Calculate the portfolio effect for a company with 5 fabs in water-stressed regions.

Case Study 24: Climate Migration and Real Estate Markets

Problem Statement: Climate change is driving migration from high-risk coastal areas to inland cities. Model the impact on real estate prices in both origin and destination markets.

Mathematical Framework:

217. **Migration Model:**

218. $\text{Migration}_{rate}(t) = M_0 * [\text{Risk}_{origin}(t) / \text{Risk}_{destination}(t)]^\gamma$

    Where $\gamma = 0.5$ (elasticity of migration to risk differential).

219. **Risk Indices:**

220. $\text{Risk}_{coastal}(t) = \alpha_{SLR} * \text{SLR}(t) + \alpha_{hurricane} * \text{Hurricane}_{intensity}(t)$ $\text{Risk}_{inland}(t) = \alpha_{heat} * \text{Heat}_{days}(t)$

221. **Housing Demand:**

222. $Demand_{coastal}(t) = Demand_0 * (1 - Migration_{rate}(t))$ $Demand_{inland}(t) = Demand_0 * (1 + Migration_{rate}(t) * Population_{ratio})$

223. **Price Dynamics:**

224. $P(t+1) = P(t) * [Demand(t) / Supply]^\epsilon$

  Where $\epsilon = 0.3$ (price elasticity).

225. **Portfolio Impact:** Investor holds:

  1. 60% coastal properties (current value $600M)
  2. 40% inland properties (current value $400M)

226. $Portfolio_{value}(t) = 0.6 * P_{coastal}(t) + 0.4 * P_{inland}(t)$

Required Analysis: - Simulate migration rates under RCP4.5 and RCP8.5 - Calculate price trajectories for both markets (30 years) - Determine optimal portfolio rebalancing strategy - Calculate the cost of inaction (maintaining 60/40 allocation vs. optimal)

Extension: Add transaction costs (5% of value) and capital gains taxes (20%). Determine the optimal rebalancing frequency and thresholds.

Case Study 25: Integrated Assessment of Climate Risk for a Diversified Conglomerate

Problem Statement: A conglomerate has operations in: - Energy (30% of value): Oil & gas production - Manufacturing (25%): Automotive parts - Real Estate (20%): Commercial properties - Agriculture (15%): Food processing - Finance (10%): Insurance and lending

Conduct a comprehensive climate risk assessment across all divisions.

Mathematical Framework:

227. **Division-Specific Risk Models:**

228. **Energy:**

  $V_{energy}(t) = \sum CF_{energy}(s) / (1 + r + \beta_{energy} * P_{carbon}(s))^s$

  Manufacturing:

$V_{manuf}(t) = \sum CF_{manuf}(s) * (1 - D_{supply\_chain}(T(s))) / (1+r)^s$

Real Estate:

$V_{RE}(t) = \sum NOI(s) * (1 - D_{physical}(T(s))) / (1+r)^s$

Agriculture:

$V_{ag}(t) = \sum CF_{ag}(s) * (1 + \beta_1(T(s)-T_0) + \beta_2(T(s)-T_0)^2) / (1+r)^s$

Finance:

$V_{fin}(t) = \sum (Premiums(s) - Losses(s, T(s))) / (1+r)^s$

229. **Correlation Structure:** Model correlations between divisions:
230. $\Sigma = [\rho_{ij}]$ where $\rho_{ij} = Corr(V_i, V_j)$
231. **Conglomerate Value:**
232. $V_{total} = \sum w_i * V_i$
233. **Climate VaR:**
234. $VaR_{95}\% = V_{baseline} - V_{5th\_percentile}$
235. **Diversification Benefit:**
236. $Benefit = \sum VaR_i - VaR_{portfolio}$

Required Analysis: - Model each division's value under three NGFS scenarios - Calculate division-specific VaRs - Estimate correlation matrix (use historical data + climate adjustments) - Calculate portfolio-level Climate VaR - Quantify diversification benefit - Identify which division contributes most to portfolio risk

Extension: Determine the optimal divestment strategy. If the conglomerate must reduce climate risk by 40%, which division(s) should be sold to maximize remaining value?

Supplementary Advanced Problems

Problem 26: Dynamic Hedging of Climate Risk with Derivatives

Design a hedging strategy using weather derivatives and carbon price futures to minimize the climate risk of an agricultural portfolio. Derive the optimal hedge ratio and calculate the hedging effectiveness.

Problem 27: Climate Risk in Mergers & Acquisitions

A company is acquiring a target with significant climate risk exposure. Develop a framework to adjust the acquisition price based on climate risk. Include earnout provisions tied to climate outcomes.

Problem 28: Optimal Climate Disclosure Strategy

Model the trade-off between transparency (full climate risk disclosure) and strategic ambiguity. Use game theory to determine the Nash equilibrium disclosure level in a competitive market.

Problem 29: Central Bank Climate Stress Testing Methodology

Design a comprehensive stress testing framework for a central bank. Include scenario generation, transmission mechanisms, and systemic risk amplification.

Problem 30: Climate Risk Transfer through Insurance-Linked Securities

Price a portfolio of catastrophe bonds, sidecars, and industry loss warranties. Optimize the capital structure to minimize cost of risk transfer.

Chapter 10: Survey of Existing Climate-Economic Models

This chapter provides a comprehensive mathematical treatment of the major climate-economic models used in research and practice. Each model is presented with its complete mathematical structure, key assumptions, calibration parameters, and applications to financial risk assessment.

10.1 The DICE Model (Dynamic Integrated Climate-Economy)

10.1.1 Model Overview

The DICE model, developed by William Nordhaus (2017 Nobel laureate), is the most influential integrated assessment model (IAM) for climate-economic analysis. It combines a Ramsey-Cass-Koopmans neoclassical growth model with a simplified climate module.

10.1.2 Mathematical Structure

Economic Module:

The economy is represented by a Cobb-Douglas production function with climate damages:

```
Q(t) = Ω(t) * A(t) * K(t)^γ * L(t)^(1-γ)
```

Where: - **Q(t)** = Gross output at time t - **Ω(t)** = Damage function (fraction of output remaining after climate damages) - **A(t)** = Total factor productivity - **K(t)** = Capital stock - **L(t)** = Labor force (population) - γ = Capital share of output (typically 0.30)

Damage Function:

The quadratic damage function relates temperature to economic damages:

```
Ω(T) = 1 / (1 + π₁*T + π₂*T²)
```

Calibration (DICE-2016R2): - $\pi_1 = 0$ - $\pi_2 = 0.00236$

This implies: - 2°C warming → 0.9% GDP loss - 3°C warming → 2.1% GDP loss - 5°C warming → 5.5% GDP loss

Capital Accumulation:

$K(t+1) = (1 - \delta_K) * K(t) + I(t)$

Where $\delta_K = 0.10$ (10% annual depreciation).

Resource Constraint:

```
Q(t) = C(t) + I(t) + Λ(t) * Q(t)
```

Where: - **C(t)** = Consumption - **I(t)** = Investment - **Λ(t)** = Abatement cost function

Abatement Cost Function:

```
Λ(μ, t) = θ₁(t) * μ(t)^θ₂
```

Where: - **μ(t)** = Emissions control rate (0 = no control, 1 = full control) - **θ_1(t)** = Cost coefficient (declining over time due to technological progress) - θ_2 = Cost exponent (typically 2.6)

Climate Module:

DICE uses a two-layer energy balance model:

$T_{AT}(t+1) = T_{AT}(t) + \xi_1 * [F(t) - \xi_2*T_{AT}(t) - \xi_3*(T_{AT}(t) - T_{LO}(t))]$
$T_{LO}(t+1) = T_{LO}(t) + \xi_4 * [T_{AT}(t) - T_{LO}(t)]$

Where: - T_{AT} = Atmospheric temperature anomaly (°C) - T_{LO} = Lower ocean temperature anomaly (°C) - F(t) = Radiative forcing (W/m²) - ξ_1 = Speed of adjustment for atmosphere (0.1005) - ξ_2 = Feedback parameter (1.17 W/m²/°C) - ξ_3 = Heat transfer coefficient (0.088 W/m²/°C) - ξ_4 = Speed of adjustment for ocean (0.025)

Radiative Forcing:

$F(t) = F_{2x} * \log_2(M_{AT}(t) / M_{AT,1750}) + F_{EX}(t)$

Where: - F_{2x} = Forcing from doubling CO_2 (3.6813 W/m²) - $M_{AT}(t)$ = Atmospheric CO_2 concentration (GtC) - $M_{AT,1750}$ = Pre-industrial concentration (588 GtC) - $F_{EX}(t)$ = Exogenous forcing from other GHGs

Carbon Cycle:

Three-reservoir model:

$$M_{AT}(t+1) = E(t) + \varphi_{11}*M_{AT}(t) + \varphi_{21}*M_{UP}(t)$$
$$M_{UP}(t+1) = \varphi_{12}*M_{AT}(t) + \varphi_{22}*M_{UP}(t) + \varphi_{32}*M_{LO}(t)$$
$$M_{LO}(t+1) = \varphi_{23}*M_{UP}(t) + \varphi_{33}*M_{LO}(t)$$

Where: - M_{AT} = Atmospheric carbon (GtC) - M_{UP} = Upper ocean/biosphere carbon (GtC) - M_{LO} = Deep ocean carbon (GtC) - E(t) = Industrial emissions (GtC/year) - φ_{ij} = Transfer coefficients (calibrated to carbon cycle models)

Emissions:

$$E(t) = \sigma(t) * (1 - \mu(t)) * Q(t) + E_{land}(t)$$

Where: - σ(t) = Emissions intensity (tCO_2/$ of output, declining over time) - $E_{land}(t)$ = Exogenous land-use emissions

Objective Function:

The social planner maximizes discounted utility:

```
W = ∑(t=0 to T) [U(C(t), L(t)) / (1+ρ)^t]
```

Where:

```
U(C, L) = L * [C/L]^(1-α) / (1-α)
```

(a) ρ = Pure rate of time preference (0.015)

(b) α = Elasticity of marginal utility (1.45)

Optimal Control:

The model solves for optimal paths of μ(t) and s(t) = I(t)/Q(t) (savings rate) that maximize W subject to all constraints.

10.1.3 Social Cost of Carbon

The **Social Cost of Carbon (SCC)** is the marginal damage from an additional ton of CO_2:

```
SCC(t) = -∂W/∂E(t) = ∑(s=t to T) [∂Q(s)/∂E(t)] * [∂U/∂C(s)] / (1+ρ)^(s-t)
```

DICE-2016R2 Calibration: - SCC(2020) ≈ \$37/tCO_2 (at 3% discount rate) - SCC grows at approximately 2-3% per year

10.1.4 Application to Financial Risk

Asset Valuation:

For a firm with emissions E_{firm}, the climate-adjusted value is:

$V_{climate} = \sum(t=1 \text{ to } T) [CF_t - SCC(t)*E_{firm}(t)] / (1+r)^t$

Stranded Asset Calculation:

Assets are stranded when carbon price exceeds profitability threshold:

$P_{carbon}(t) > (Revenue - OpEx) / Emissions$

Using DICE's optimal carbon price path, determine the stranding date t*.

10.2 The FUND Model (Climate Framework for Uncertainty, Negotiation and Distribution)

10.2.1 Model Overview

FUND, developed by Richard Tol, is a disaggregated IAM with detailed sectoral and regional damage functions. It emphasizes uncertainty quantification through Monte Carlo analysis.

10.2.2 Mathematical Structure

Regional Structure:

FUND divides the world into 16 regions, each with its own economic and climate module.

Damage Function:

Unlike DICE's aggregate function, FUND models damages by sector:

$D_{total,r}(t) = \sum(sectors)\ D_{sector,r}(T_r(t),\ Y_r(t),\ t)$

Sectoral Damage Functions:

237. **Agriculture:**

238. $D_{ag,r} = \alpha_{ag,r} * T_r + \beta_{ag,r} * T_{r^2} + \gamma_{ag,r} * Y_r * T_r$

Where Y_r is income per capita in region r.

239. **Sea Level Rise:**

240. $D_{SLR,r} = \alpha_{SLR,r} * (SLR(t) / (1 + \beta_{SLR,r} * SLR(t)))$

With SLR modeled as:

$dSLR/dt = \alpha_{thermal} * T + \alpha_{ice} * \max(0, T - T_{threshold})$

241. **Health (Heat Mortality):**

242. $D_{health,r} = \alpha_{health,r} * Population_r * (T_r - T_{optimal,r})^2 / Y_r\text{^}\epsilon$

Where $\epsilon = 0.5$ (income elasticity of adaptation).

243. **Energy (Cooling/Heating):**

244. $D_{energy,r} = \alpha_{cool,r} * CDD(T_r) - \alpha_{heat,r} * HDD(T_r)$

Where CDD = cooling degree days, HDD = heating degree days.

245. **Ecosystems:**

246. $D_{eco,r} = \alpha_{eco,r} * (1 - \exp(-\beta_{eco,r} * T_r))$

247. **Extreme Weather:**

248. $D_{extreme,r} = \alpha_{extreme,r} * T_r\text{^}\gamma_{extreme}$

With $\gamma_{extreme} \approx 2\text{-}3$ (super-linear relationship).

Total Damages:

$D_{total}(t) = \sum(r=1 \text{ to } 16)\ w_r * D_{total,r}(t)$

Where w_r is the economic weight of region r (typically GDP share).

Uncertainty Quantification:

FUND specifies probability distributions for key parameters:

Parameter	Distribution	Mean	Std Dev
Climate Sensitivity	Log-normal	3.0°C	1.5°C
α_{ag}	Normal	-0.04	0.02
β_{ag}	Normal	-0.0014	0.0007
Discount rate	Triangular	1.0%	0.5%

Monte Carlo Simulation:

Run N simulations (typically 10,000):

For i = 1 to N:
  Draw parameters from distributions
  Solve model → SCC_i
End

SCC_{mean} = mean(SCC_i)
SCC_{95}% = 95th percentile(SCC_i)

10.2.3 Equity Weighting

FUND allows for equity weighting across regions:

$W_{equity} = \sum(r) \sum(t) [U_r(C_r(t)) * (Y_{ref} / Y_r(t))^\eta] / (1+\rho)^t$

Where: - η = Equity weight parameter (0 = no weighting, 1 = full weighting) - Y_{ref} = Reference income level

This increases the weight on damages in poor regions.

10.2.4 Application to Financial Risk

Sectoral Risk Assessment:

For a portfolio with exposures to different sectors:

$Risk_{portfolio} = \sum(\text{sectors})\ w_{sector} * E[D_{sector}]$

Regional Diversification:

Calculate correlation matrix of regional damages:

$Corr(D_r, D_s) = Cov(D_r, D_s) / (\sigma_r * \sigma_s)$

Use this to optimize regional portfolio allocation.

10.3 The PAGE Model (Policy Analysis of the Greenhouse Effect)

10.3.1 Model Overview

PAGE, developed by Chris Hope at Cambridge, emphasizes fat-tailed risk and discontinuities (tipping points). It uses probabilistic rather than deterministic modeling.

10.3.2 Mathematical Structure

Damage Function:

PAGE uses a power function with regional variation:

$D_r(T) = (a_r / T_{tol,r}) * T^{\wedge}b_r$

Where: - $T_{tol,r}$ = Tolerable temperature for region r (triangular distribution: 2-3°C) - a_r = Damage coefficient (triangular: 0.5-2.5%) - b_r = Damage exponent (triangular: 1.5-3.0)

Discontinuity (Tipping Point):

Additional damage if temperature exceeds threshold:

$D_{discontinuity} = D_{disc} * I(T > T_{threshold})$

Where: - $I(\cdot)$ = Indicator function - $T_{threshold}$ ~ Uniform(2.5, 4.5°C) - D_{disc} ~ Uniform(5%, 25% of GDP)

Total Damage:

$D_{total} = D_{continuous} + D_{discontinuity}$

Climate Sensitivity:

PAGE uses a fat-tailed distribution:

```
CS ~ Log-normal(μ = log(2.5), σ = 0.5)
```

This gives: - Median CS = 2.5°C - 95th percentile CS ≈ 6°C - Long right tail (captures low-probability, high-impact outcomes)

Discounting:

PAGE allows for time-varying discount rates:

$r(t) = r_0 * \exp(-\lambda * t)$

Where $\lambda = 0.01$ (discount rate declines over time, increasing weight on future).

10.3.3 Probability Distributions

Parameter	Distribution	Parameters
Climate Sensitivity	Log-normal	μ=log(2.5), σ=0.5
Damage exponent	Triangular	(1.5, 2.25, 3.0)
Tipping point threshold	Uniform	(2.5, 4.5°C)
Tipping point impact	Uniform	(5%, 25% GDP)
Discount rate	Triangular	(0.5%, 1.0%, 1.5%)

10.3.4 Expected Value Calculation

$E[SCC] = \int\int\int SCC(CS, b, T_{threshold}) * f(CS) * g(b) * h(T_{threshold})\, dCS\, db\, dT_{threshold}$

Computed via Monte Carlo:

$SCC_{mean} = (1/N) * \sum(i=1 \text{ to } N)\, SCC_i$

PAGE Results:

(a) Mean SCC(2020) ≈ \$100/tCO_2 (higher than DICE due to fat tails)

(b) 95th percentile SCC ≈ \$400/tCO_2

(c) Strong sensitivity to discount rate and tipping point parameters

10.3.5 Application to Financial Risk

Tail Risk Measurement:

PAGE is ideal for calculating tail risk metrics:

$CVaR_{95}\% = E[Loss \mid Loss > VaR_{95}\%]$

Scenario Analysis:

Generate scenarios from PAGE distributions: - Optimistic (10th percentile): CS=1.8°C, no tipping, SCC=\$30 - Base (50th percentile): CS=2.5°C, no tipping, SCC=\$100 - Pessimistic (90th percentile): CS=4.5°C, tipping, SCC=\$300

10.4 The REMIND Model (Regional Model of Investments and Development)

10.4.1 Model Overview

REMIND, developed by PIK Potsdam, is a technology-rich IAM focusing on energy system transformation. It uses intertemporal optimization to find cost-minimizing pathways to climate targets.

10.4.2 Mathematical Structure

Production Function:

Nested CES (Constant Elasticity of Substitution) structure:

$Y = [\alpha_K * K^\rho + \alpha_L * L^\rho + \alpha_E * E^\rho]^{(1/\rho)}$

Where: - **E** = Energy aggregate - ρ = Substitution parameter (related to elasticity $\sigma = 1/(1-\rho)$)

Energy Aggregate:

Further nested CES:

$E = [\alpha_{fossil} * E_{fossil}\hat{}\rho_E + \alpha_{renewable} * E_{renewable}\hat{}\rho_E]\hat{}(1/\rho_E)$

Technology Portfolio:

Energy is produced by a portfolio of technologies:

$E_{fossil} = \sum(\text{tech} \in \{\text{coal, gas, oil}\})\ E_{tech}$
$E_{renewable} = \sum(\text{tech} \in \{\text{solar, wind, hydro, nuclear}\})\ E_{tech}$

$¿kW¿-¿*OPEX_tech*¿=Operatingcost¿$Each technology has: - $CAPEX_{tech}$ = Capital cost (/MWh)
- $Emissions_{tech}$ = Emissions intensity (tCO_2/MWh) - $Capacity_{factor\_tech}$ = Utilization rate

Optimization Problem:

$\min \sum(t=1 \text{ to } T) \sum(\text{tech}) [CAPEX_{tech} * NewCapacity_{tech}(t) + OPEX_{tech} * Generation_{tech}(t)] / (1+r)\hat{}t$

subject to:
- Energy demand met: $\sum(\text{tech})\ Generation_{tech}(t) \geq Demand(t)$
- Capacity constraint: $Generation_{tech}(t) \leq Capacity_{tech}(t) * CF_{tech}$
- Capacity evolution: $Capacity_{tech}(t+1) = (1-\delta_{tech})*Capacity_{tech}(t) + NewCapacity_{tech}(t)$
- Emissions constraint: $\sum(\text{tech})\ Emissions_{tech} * Generation_{tech}(t) \leq Budget(t)$
- Non-negativity: All variables ≥ 0

Carbon Budget:

For a 2°C target:

$\sum(t=2020 \text{ to } 2100)\ Emissions(t) \leq Budget_{2C} \approx 1{,}000\ GtCO_2$

Endogenous Technical Change:

Technology costs decline with cumulative deployment (learning-by-doing):

$CAPEX_{tech}(t) = CAPEX_{tech,0} * (CumulativeCapacity_{tech}(t) / CumulativeCapacity_{tech,0})\hat{}(-LR_{tech})$

Where: - LR_{tech} = Learning rate (typically 10-20% for solar, wind)

10.4.3 REMIND Calibration

Technology	CAPEX (2020)	Learning Rate	Emissions
Coal	$2,000/kW	5%	0.9 tCO_2/MWh
Gas	$1,000/kW	5%	0.4 tCO_2/MWh
Solar PV	$1,200/kW	20%	0
Wind	$1,500/kW	15%	0
Nuclear	$5,000/kW	2%	0

10.4.4 Solution Method

REMIND is formulated as a Mixed Complementarity Problem (MCP) and solved using PATH solver. The solution provides: - Optimal technology deployment schedule - Shadow price of carbon (marginal cost of emissions reduction) - Total system cost

10.4.5 Application to Financial Risk

Stranded Asset Identification:

Technologies are stranded when their levelized cost exceeds the market price:

$$LCOE_{tech}(t) = [CAPEX_{tech} * CRF + OPEX_{tech} + Emissions_{tech} * P_{carbon}(t)] / CF_{tech}$$

Stranded if: $LCOE_{tech}(t) > P_{electricity}(t)$

Transition Risk Quantification:

For a utility with capacity mix $\{Cap_{coal}, Cap_{gas}, Cap_{renewable}\}$:

$$TransitionRisk = \sum(tech)\ Cap_{tech} * \max(0, LCOE_{tech} - P_{electricity}) * Lifetime_{tech}$$

10.5 NGFS Climate Scenarios

10.5.1 Scenario Framework

The Network for Greening the Financial System (NGFS) provides standardized scenarios for financial sector stress testing. These scenarios are generated using multiple IAMs (primarily REMIND, MESSAGE, GCAM).

10.5.2 Scenario Typology

Orderly Scenarios: 1. **Net Zero 2050:** Immediate policy action, limiting warming to 1.5°C - Carbon price: \$100/tCO_2 (2030) → \$600/tCO_2 (2050) - Renewable share: 70% by 2050 - GDP impact: -1% to -3% vs. baseline

2. **Below 2°C:** Gradual transition, limiting warming to 1.7°C
    1. Carbon price: \$50/tCO_2 (2030) → \$300/tCO_2 (2050)
    2. Renewable share: 60% by 2050
    3. GDP impact: -0.5% to -2% vs. baseline

Disorderly Scenarios: 3. **Delayed Transition:** Policy action delayed until 2030, then rapid catch-up - Carbon price: \$0 (2020-2030), then \$200/tCO_2 (2035) → \$1,000/tCO_2 (2050) - Renewable share: 75% by 2050 (rapid deployment) - GDP impact: -3% to -5% vs. baseline (higher transition costs)

4. **Divergent Net Zero:** Uncoordinated policies across regions
    1. Carbon price varies by region: \$50-\$500/tCO_2 (2050)
    2. Trade tensions, carbon border adjustments
    3. GDP impact: -2% to -4% vs. baseline

Hot House World: 5. **NDCs:** Only current nationally determined contributions implemented - Carbon price: \$25/tCO_2 (2050) (weak policy) - Warming: 2.5-3.0°C by 2100 - Physical damages: -5% to -10% GDP by 2100

6. **Current Policies:** No new policies beyond those already in place
    1. Carbon price: \$10/tCO_2 (2050)

2. Warming: 3.0-3.5°C by 2100
3. Physical damages: -10% to -20% GDP by 2100

10.5.3 Mathematical Specification

Carbon Price Path (Net Zero 2050):

$P_{carbon}(t) = P_0 * \exp(r_{carbon} * (t - 2020))$

Where: - P_0 = \$75/tCO_2 (2020) - $r_{carbon} = 0.08$ (8% annual growth)

Temperature Path:

$T(t) = T_{2020} + \Delta T_{scenario} * [1 - \exp(-\lambda * (t - 2020))]$

Where: - T_{2020} = 1.2°C - $\Delta T_{scenario}$ = 0.3°C (Net Zero), 0.5°C (Below 2°C), 1.8°C (Current Policies) - $\lambda = 0.02$ (convergence rate)

GDP Impact:

$GDP(t) = GDP_{baseline}(t) * (1 - D_{transition}(t) - D_{physical}(t))$

Where: - $D_{transition}(t) = \alpha_{trans} * P_{carbon}(t) * Carbon_{intensity}(t)$ - $D_{physical}(t) = \beta_{phys} * T(t)^2$

10.5.4 Application to Financial Stress Testing

Bank Loan Portfolio:

For each sector s:

$PD_s(t) = PD_{baseline,s} * \exp(\beta_{s,trans} * P_{carbon}(t) + \beta_{s,phys} * T(t))$

Expected Loss:

$EL(t) = \sum(s)\ EAD_s * PD_s(t) * LGD_s$

Capital Requirement:

$Capital_{required}(t) = \max(s \in scenarios)\ EL_s(t) * 1.5$

10.6 Comparison of Models

10.6.1 Key Differences

Feature	DICE	FUND	PAGE	REMIND
Regions	1 (global)	16	8	21
Sectors	Aggregate	8	Aggregate	Energy-detailed
Damage Function	Quadratic	Sectoral	Power + Tipping	Exogenous
Uncertainty	Deterministic	Monte Carlo	Monte Carlo	Deterministic
Time Horizon	2100	2300	2200	2100
Time Step	5 years	1 year	1 year	5 years
Climate Module	2-box	2-box	1-box	Exogenous
SCC (2020)	\$37/tCO_2	\$50/tCO_2	\$100/tCO_2	N/A (policy-driven)

10.6.2 Model Selection Guidance

Use DICE when: - Need simple, transparent model - Focus on optimal policy (carbon tax) - Global aggregate analysis sufficient

Use FUND when: - Regional/sectoral detail required - Uncertainty quantification critical - Equity considerations important

Use PAGE when: - Tail risk and tipping points key concern - Fat-tailed distributions needed - Precautionary approach justified

Use REMIND when: - Energy system transformation focus - Technology portfolio optimization - Feasibility of climate targets

Use NGFS scenarios when: - Financial sector stress testing - Standardized scenarios required - Regulatory compliance (central banks)

10.7 Advanced Topics

10.7.1 Meta-Analysis of IAMs

Combining results from multiple models:

$SCC_{ensemble} = \sum(m=1 \text{ to } M)\ w_m * SCC_m$

Where weights w_m can be: - Equal weights: $w_m = 1/M$ - Performance-based: $w_m \propto (1 / RMSE_m)$ - Bayesian: $w_m \propto P(\text{Data} \mid \text{Model}_m)$

10.7.2 Emulators and Surrogate Models

For computational efficiency, build statistical emulators:

$SCC \approx f(CS, \text{Discount}_{rate}, \text{Damage}_{parameters}) + \epsilon$

Using Gaussian Process regression or neural networks.

10.7.3 Recursive Dynamic Equilibrium

Some advanced IAMs (e.g., WITCH) solve for Nash equilibrium in a game-theoretic framework:

$\max_{\mu\_r} W_r(\mu_r, \mu_{-r})$

Where each region r optimizes its own welfare given other regions' strategies.

10.8 Supplementary Problems

249. **Calibrate DICE damage function** to match the IPCC AR6 damage estimates. What values of π_1 and π_2 are required?

250. **Decompose FUND's SCC** into contributions from each damage sector. Which sector dominates?

251. **Calculate the probability** in PAGE that damages exceed 10% of GDP by 2100.

252. **Solve REMIND** for a 1.5°C target with and without carbon capture and storage (CCS). How much does CCS reduce total system cost?

253. **Compare NGFS scenarios:** Calculate the stranded asset value for a coal power plant under "Net Zero 2050" vs. "Current Policies."

254. **Sensitivity analysis:** How does DICE's SCC change when climate sensitivity increases from 3.0°C to 4.5°C?

255. **Regional equity:** Using FUND, calculate the SCC with equity weighting (η=1) vs. without (η=0). How much higher is the equity-weighted SCC?

256. **Tipping point impact:** In PAGE, what is the expected value of damages conditional on a tipping point occurring?

257. **Technology learning:** In REMIND, how much does solar PV cost decline if cumulative deployment doubles? Triples?

258. **Model ensemble:** Calculate a weighted average SCC using DICE (weight 0.3), FUND (weight 0.4), and PAGE (weight 0.3). Justify the weights.

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Appendices

Appendix A: Mathematical Notation and Conventions

A.1 General Notation

Scalars: - Lowercase italic letters: *t* (time), *r* (rate), *T* (temperature) - Greek letters: α, β, γ, δ, ϵ, λ, μ, σ, ρ, τ

Vectors: - Lowercase bold letters: x, μ, β - Dimension indicated by subscript when necessary: x_n ∈ $\mathbb{R}^n$

Matrices: - Uppercase bold letters: **A, Σ, Ω** - Identity matrix: **I** - Transpose: $\mathbf{A}^T$ or **A'**

Random Variables: - Uppercase italic letters: X, Y, Z - Stochastic processes: W_t, B_t (Brownian motion)

A.2 Operators and Functions

Statistical Operators: - E[·] = Expectation operator - Var[·] = Variance operator - Cov[·,·] = Covariance operator - Corr[·,·] = Correlation coefficient - Pr(·) or P(·) = Probability - Φ(·) = Standard normal cumulative distribution function - φ(·) = Standard normal probability density function

Calculus: - ∂f/∂x = Partial derivative of f with respect to x - df/dx = Total derivative - ∇f = Gradient vector - ∫ = Integral - ∑ = Summation - lim = Limit

Special Functions: - ln(·) = Natural logarithm (base e) - log(·) = Logarithm (base 10 unless specified) - $\log_2(\cdot)$ = Logarithm base 2 - exp(·) = Exponential function (e^·) - max(·,·) = Maximum - min(·,·) = Minimum

A.3 Financial Variables

Symbol	Description	Units
V	Asset value	$ or local currency

CF	Cash flow	$ or local currency
r	Discount rate / risk-free rate	Decimal (e.g., 0.05 = 5%)
r_f	Risk-free rate	Decimal
WACC	Weighted average cost of capital	Decimal
NPV	Net present value	$
IRR	Internal rate of return	Decimal
PV	Present value	$
FV	Future value	$
Q	Output / production	$ or physical units
K	Capital stock	$
I	Investment	$
C	Consumption	$
Y	Income / GDP	$
L	Labor / population	Persons or person-years
A	Total factor productivity	Dimensionless

A.4 Risk Metrics

Symbol	Description	Units
VaR_α	Value-at-Risk at confidence level α	$
ES_α or $CVaR_\alpha$	Expected Shortfall / Conditional VaR	$
σ	Standard deviation / volatility	$ or %
σ^2	Variance	$² or %²
β	Beta (systematic risk)	Dimensionless
α	Alpha (excess return)	% or $
ρ	Correlation coefficient	Dimensionless [-1,1]

A.5 Climate Variables

Symbol	Description	Units
T	Temperature anomaly	°C above pre-industrial
T_{AT}	Atmospheric temperature	°C
T_{LO}	Lower ocean temperature	°C
F	Radiative forcing	W/m^2
F_{2x}	Forcing from CO_2 doubling	W/m^2
λ	Climate feedback parameter	$W/m^2/K$
ECS	Equilibrium climate sensitivity	°C
TCR	Transient climate response	°C
M	Carbon mass / concentration	GtC or ppm
M_{atm}	Atmospheric carbon	GtC
C	CO_2 concentration	ppm
E	Emissions	$GtCO_2$/year or GtC/year
SLR	Sea level rise	meters
P	Precipitation	mm/year

A.6 Damage Functions

Symbol	Description	Units
D(T)	Damage function	Fraction of GDP lost
Ω(T)	Output remaining after damages	Dimensionless [0,1]
π	Damage coefficient	Various
β	Temperature sensitivity	Various

A.7 Stochastic Processes

Brownian Motion: - W_t or B_t = Standard Brownian motion - dW_t = Increment of Brownian motion - $E[dW_t] = 0$ - $Var[dW_t] = dt$

Stochastic Differential Equations: - $dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$ - μ = Drift coefficient - σ = Diffusion coefficient

Geometric Brownian Motion: - $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$ - S_t = Asset price at time t

A.8 Probability Distributions

Normal Distribution: - $X \sim N(\mu, \sigma^2)$ - PDF: $f(x) = (1/(\sigma\sqrt{(2\pi)}))\ \exp(-(x-\mu)^2/(2\sigma^2))$

Log-Normal Distribution: - $X \sim LN(\mu, \sigma^2)$ - If $\ln(X) \sim N(\mu, \sigma^2)$, then $X \sim LN(\mu, \sigma^2)$

Uniform Distribution: - $X \sim U(a, b)$ - PDF: $f(x) = 1/(b-a)$ for $x \in [a,b]$

Triangular Distribution: - $X \sim Tri(a, b, c)$ where $a \le b \le c$ - Mode at b

Pareto Distribution: - $X \sim Pareto(\alpha, x_m)$ - Used for extreme events / tail risk

Appendix B: Probability Distributions Used in Climate Finance

B.1 Normal Distribution

Application: General uncertainty in parameters, returns, errors

Parameters: - μ = mean - σ^2 = variance

Properties: - Symmetric around mean - 68% of mass within $\pm 1\sigma$ - 95% within $\pm 1.96\sigma$ - 99% within $\pm 2.58\sigma$

Generation: Box-Muller transform or inverse CDF method

B.2 Log-Normal Distribution

Application: Asset prices, positive-only variables

Parameters: - μ = mean of log(X) - σ^2 = variance of log(X)

Properties: - Always positive - Right-skewed - $E[X] = \exp(\mu + \sigma^2/2)$ - $Var[X] = (\exp(\sigma^2) - 1) \exp(2\mu + \sigma^2)$

Generation: $X = \exp(Y)$ where $Y \sim N(\mu, \sigma^2)$

B.3 Triangular Distribution

Application: Expert elicitation, scenario analysis

Parameters: - a = minimum - b = mode (most likely) - c = maximum

Properties: - Simple to specify (min, mode, max) - Mean = $(a + b + c)/3$ - Used in PAGE model for damage parameters

B.4 Uniform Distribution

Application: Complete uncertainty, tipping point thresholds

Parameters: - a = lower bound - b = upper bound

Properties: - All values equally likely - Mean = $(a + b)/2$ - Variance = $(b - a)^2/12$

B.5 Pareto Distribution

Application: Extreme events, catastrophe losses

Parameters: - α = shape parameter (tail index) - x_m = scale parameter (minimum value)

Properties: - Heavy-tailed ($\alpha < 2 \rightarrow$ infinite variance) - Power law: $P(X > x) \propto x^{-\alpha}$ - Used in insurance for large losses

PDF: $f(x) = (\alpha x_m^\alpha) / x^{(\alpha+1)}$ for $x \geq x_m$

B.6 Compound Poisson Distribution

Application: Aggregate losses from multiple events

Structure: - $N \sim \text{Poisson}(\lambda)$ = number of events - $X_i \sim F$ = severity of each event - $S = \sum_{i=1}^{N} X_i$ = total loss

Properties: - $E[S] = \lambda E[X]$ - $Var[S] = \lambda E[X^2]$ - Used for hurricane losses, operational risk

Appendix C: Numerical Methods and Algorithms

C.1 Monte Carlo Simulation Algorithm

Algorithm: Climate Value-at-Risk Calculation

```
Input:
  - N = number of simulations
  - T = time horizon (years)
  - Parameter distributions

Output:
  - VaR_α = Value-at-Risk at confidence level α
  - ES_α = Expected Shortfall

Procedure:
  1. Initialize results array: V[1..N]

  2. For i = 1 to N:
       a. Draw climate parameters:
            - CS ~ LogNormal(μ_CS, σ_CS)   // Climate sensitivity
            - λ ~ Normal(μ_λ, σ_λ)          // Feedback parameter

       b. Draw economic parameters:
            - β ~ Normal(μ_β, σ_β)          // Damage coefficient
            - r ~ Uniform(r_min, r_max)     // Discount rate

       c. Simulate temperature path:
            For t = 1 to T:
                T[t] = T[t-1] + (F[t]/λ - T[t-1])/τ + σ_T * √(Δt) * Z[t]
```

where $Z[t] \sim N(0,1)$

d. Calculate damages:

For t = 1 to T:

$D[t] = \beta_1 * T[t] + \beta_2 * T[t]^2$

e. Compute cash flows:

For t = 1 to T:

$CF[t] = CF_{baseline}[t] * (1 - D[t])$

f. Calculate present value:

$V[i] = \sum(t=1 \text{ to } T) \; CF[t] / (1+r)^t$

3. Sort results: $V_{sorted} = \text{sort}(V)$

4. Calculate VaR:

$index_{VaR} = \text{floor}((1-\alpha) * N)$

$VaR_\alpha = V_{baseline} - V_{sorted}[index_{VaR}]$

5. Calculate ES:

$ES_\alpha = V_{baseline} - \text{mean}(V_{sorted}[1..index_{VaR}])$

6. Return VaR_α, ES_α

C.2 Finite Difference Method for PDEs

Algorithm: Implicit Finite Difference for Climate-Adjusted Option Pricing

Input:

- S_{max} = maximum asset price
- T_{max} = time to maturity

- N_S = number of space steps
- N_T = number of time steps
- $\sigma(t)$ = time-dependent volatility
- r = risk-free rate
- K = strike price

Output:

- V(S,0) = option value today

Procedure:

1. Initialize grid:

   $\Delta S = S_{max} / N_S$

   $\Delta t = T_{max} / N_T$

   $S[i] = i * \Delta S$ for $i = 0$ to N_S

   $t[j] = j * \Delta t$ for $j = 0$ to N_T

2. Set terminal condition (European call):

   For $i = 0$ to N_S:

      $V[i, N_T] = \max(S[i] - K, 0)$

3. Set boundary conditions:

   $V[0, j] = 0$ for all j // Out of money

   $V[N_S, j] = S_{max} - K * \exp(-r*(T_{max} - t[j]))$ // Deep in money

4. Build tridiagonal matrix for each time step:

   For $j = N_T-1$ down to 0:

      $\sigma_j = \sigma(t[j])$ // Time-dependent volatility

      For $i = 1$ to N_S-1:

         $a[i] = -0.5 * \Delta t * (\sigma_{j^2} * i^2 - r * i)$

$b[i] = 1 + \Delta t * (\sigma_{j^2} * i^2 + r)$

$c[i] = -0.5 * \Delta t * (\sigma_{j^2} * i^2 + r * i)$

Solve tridiagonal system:

A * V[:,j] = V[:,j+1]

where A is tridiagonal with diagonals (a, b, c)

5. Return V[:,0] = option values at t=0

C.3 Sensitivity Analysis: Finite Difference Approximation

Algorithm: Numerical Sensitivity (Greek) Calculation

Input:

- f(x) = function to differentiate
- x_0 = point of evaluation
- h = step size (default: 0.01)

Output:

- $df/dx|_{x=x_0}$ = numerical derivative

Procedure:

1. Central difference (most accurate):

   $df/dx \approx (f(x_0 + h) - f(x_0 - h)) / (2h)$

2. Forward difference (if $x_0 - h$ invalid):

   $df/dx \approx (f(x_0 + h) - f(x_0)) / h$

3. Second derivative (for convexity):

   $d^2f/dx^2 \approx (f(x_0 + h) - 2f(x_0) + f(x_0 - h)) / h^2$

4. For multi-dimensional sensitivity:
   $\partial f/\partial x_i \approx (f(x_0 + h*e_i) - f(x_0 - h*e_i)) / (2h)$
   where e_i is the i-th unit vector

Appendix D: Data Sources and Calibration Parameters

D.1 Climate Data Sources

Temperature Data: - NASA GISTEMP: https://data.giss.nasa.gov/gistemp/ - NOAA Global Climate Report: https://www.ncei.noaa.gov/ - Berkeley Earth: http://berkeleyearth.org/ - Hadley Centre (HadCRUT5): https://www.metoffice.gov.uk/hadobs/hadcrut5/

CO_2 Concentration: - Mauna Loa Observatory: https://gml.noaa.gov/ccgg/trends/ - Global Carbon Project: https://www.globalcarbonproject.org/ - NOAA GML: https://gml.noaa.gov/

Sea Level: - NOAA Sea Level Trends: https://tidesandcurrents.noaa.gov/sltrends/ - NASA Sea Level Portal: https://sealevel.nasa.gov/ - CSIRO: https://www.cmar.csiro.au/sealevel/

Climate Projections: - CMIP6 (Coupled Model Intercomparison Project): https://esgf-node.llnl.gov/projects/cmip6/ - IPCC Data Distribution Centre: https://www.ipcc-data.org/

D.2 Economic and Financial Data

GDP and Macroeconomic: - World Bank World Development Indicators: https://databank.worldbank.org/ - IMF World Economic Outlook: https://www.imf.org/en/Publications/WEO - OECD Statistics: https://stats.oecd.org/ - Penn World Table: https://www.rug.nl/ggdc/productivity/pwt/

Financial Markets: - Bloomberg Terminal - Refinitiv Eikon - Yahoo Finance: https://finance.yahoo.com/ - FRED (Federal Reserve Economic Data): https://fred.stlouisfed.org/

Carbon Prices: - World Bank Carbon Pricing Dashboard: https://carbonpricingdashboard.worldbank.org/ - EU ETS: https://ember-climate.org/data/carbon-price-viewer/ - ICAP (International Carbon Action Partnership): https://icapcarbonaction.com/

D.3 Calibrated Parameters for Models

Table D.1: Climate Physics Parameters

Parameter	Symbol	Value	Source
Forcing from CO_2 doubling	F_{2x}	3.71 W/m²	IPCC AR6 [3]
Climate feedback parameter	λ	1.1 W/m²/K	IPCC AR6 [3]
Equilibrium climate sensitivity	ECS	3.0°C (2.5-4.0)	IPCC AR6 [3]
Transient climate response	TCR	1.8°C (1.4-2.2)	IPCC AR6 [3]
Pre-industrial CO_2	C_0	280 ppm	IPCC AR6 [3]
Current CO_2 (2023)	C	420 ppm	NOAA [10]
Airborne fraction	AF	0.44	Global Carbon Budget [10]

Table D.2: Economic Damage Function Parameters

Model	Parameter	Value	Source
BHM	β_1	0.0127	Burke et al. (2015) [13]
BHM	β_2	-0.0005	Burke et al. (2015) [13]
BHM	$T_{optimal}$	13°C	Burke et al. (2015) [13]
DICE-2016R2	π_2	0.00236	Nordhaus (2017) [19]
FUND	α_{ag}	-0.04	Tol (2002) [23]
FUND	β_{ag}	-0.0014	Tol (2002) [23]

Table D.3: Financial Parameters

Parameter	Symbol	Typical Value	Range
Risk-free rate	r_f	2-3%	0-5%
Equity risk premium	ERP	5-7%	3-10%
Discount rate (social)	ρ	1.5%	0.5-3%
Elasticity of marginal utility	η	1.5	1.0-2.0
Corporate WACC	WACC	8-10%	5-15%

Table D.4: NGFS Scenario Parameters (2050)

Scenario	Carbon Price ($/t$CO_2$)	Temperature (°C)	Renewable Share
Net Zero 2050	600	1.5	70%
Below 2°C	300	1.7	60%
Delayed Transition	1,000	1.8	75%
NDCs	25	2.5	40%
Current Policies	10	3.0	30%

Source: NGFS (2023) [50]

D.4 Conversion Factors

Carbon Units: - 1 GtC = 3.67 GtCO_2 - 1 ppm CO_2 ≈ 2.13 GtC - 1 tonne CO_2 = 0.273 tonnes C

Energy Units: - 1 TWh = 10^9 kWh - 1 EJ = 277.78 TWh - 1 Mtoe = 11.63 TWh

Temperature: - °C = (°F - 32) × 5/9 - K = °C + 273.15

D.5 Model Calibration Notes

Climate Sensitivity: The IPCC AR6 assessment [3] narrowed the range of ECS to 2.5-4.0°C (likely range) based on multiple lines of evidence: 1. Process understanding from climate models 2. Historical warming observations 3. Paleoclimate proxy data

For financial modeling, we recommend: - **Central estimate:** ECS = 3.0°C - **Uncertainty:** Log-normal distribution with $\sigma = 0.4$

Damage Functions: The Burke-Hsiang-Miguel (BHM) parameters [13] are estimated from historical panel data (1960-2010) for 166 countries. The quadratic form captures: - Positive effects of warming in cold countries - Negative effects in warm countries - Optimal temperature around 13°C

For financial applications: - Use BHM for country/region-specific analysis - Use DICE for global aggregate analysis - Consider both for robustness checks

Discount Rates: The choice of discount rate is contentious in climate economics [32]. We recommend: - **Market discount rate (private sector):** Use WACC (8-10%) - **Social discount rate (policy analysis):** Use Ramsey formula: $\rho + \eta * g$ - ρ = pure time preference (1-2%) - η = elasticity of marginal utility (1.5) - g = growth rate (2%) - Yields approximately 4-5%

Carbon Prices: NGFS scenarios [50] provide carbon price trajectories. For financial modeling: - Extract prices for specific years (2030, 2050, 2100) - Interpolate using exponential growth: $P(t) = P_0 * \exp(r_{carbon} * t)$ - Estimate r_{carbon} from scenario data

Appendix E: Software and Computational Tools

E.1 Recommended Software

Statistical Computing: - R (with packages: tidyverse, ggplot2, forecast) - Python (with libraries: numpy, pandas, scipy, statsmodels) - MATLAB - Julia

Monte Carlo Simulation: - @RISK (Excel add-in) - Crystal Ball (Oracle) - Python: numpy.random, scipy.stats - R: mc2d, mc package

PDE Solvers: - MATLAB PDE Toolbox - Python: FiPy, FEniCS - R: ReacTran package

Optimization: - GAMS (General Algebraic Modeling System) - AMPL - Python: scipy.optimize, cvxpy - R: optim, nloptr

Climate Models: - DICE model: Excel version available at https://williamnordhaus.com/ - FUND model: Code at http://www.fund-model.org/ - PAGE model: Available upon request from Cambridge - REMIND: Open-source at https://github.com/remindmodel/remind

E.2 Python Code Examples

Example: Monte Carlo Climate VaR

```
import numpy as np
import matplotlib.pyplot as plt


def climate_var_simulation(N=10000, T=30, alpha=0.95):
    """
    Monte Carlo simulation for Climate Value-at-Risk

    Parameters:
    N: number of simulations
    T: time horizon (years)
    alpha: confidence level for VaR

    Returns:
    VaR, ES, loss_distribution
    """
    # Parameters
    CF_baseline = 100  # Million $ per year
    r = 0.08  # Discount rate

    # Initialize results
    PV = np.zeros(N)

    for i in range(N):
        # Draw climate parameters
```

```
        ECS = np.random.lognormal(np.log(3.0), 0.4)
        beta2 = np.random.normal(0.00236, 0.001)

        # Simulate temperature path
        T_path = np.zeros(T)
        for t in range(1, T):
            T_path[t] = T_path[t-1] + 0.05 + 0.01 * np.random.randn()

        # Calculate damages and cash flows
        CF = np.zeros(T)
        for t in range(T):
            damage = beta2 * T_path[t]**2
            CF[t] = CF_baseline * (1 - damage)

        # Calculate present value
        discount_factors = np.array([(1+r)**(-t) for t in range(1, T+1)])
        PV[i] = np.sum(CF * discount_factors)

    # Calculate baseline PV (no climate change)
    PV_baseline = CF_baseline * np.sum([(1+r)**(-t) for t in range(1, T+1)])

    # Calculate losses
    losses = PV_baseline - PV

    # Calculate VaR and ES
    VaR = np.percentile(losses, 100*alpha)
    ES = np.mean(losses[losses >= VaR])

    return VaR, ES, losses
```

```
# Run simulation
VaR_95, ES_95, losses = climate_var_simulation()
print(f"95% VaR: ${VaR_95:.2f}M")
print(f"95% ES: ${ES_95:.2f}M")

# Plot loss distribution
plt.hist(losses, bins=50, density=True, alpha=0.7)
plt.axvline(VaR_95, color='r', linestyle='--', label=f'95% VaR: ${VaR_95:.1f}M')
plt.xlabel('Loss ($M)')
plt.ylabel('Probability Density')
plt.title('Climate Value-at-Risk Distribution')
plt.legend()
plt.show()
```

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