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MATHEMATICAL NOTATION AND CONVENTIONS

Throughout this textbook, we employ rigorous mathematical notation to provide precise formulations of plastic accounting principles.

Vector and Tensor Notation

- **Scalars:** Lowercase letters p or Greek letters α
- **Vectors:** Bold lowercase letters $\mathbf{p}$
- **Matrices:** Bold uppercase letters $\mathbf{P}$
- **Tensors:** Calligraphic letters $\mathscr{P}$

Einstein Summation Convention

Repeated indices imply summation:

$\Sigma(i=1 \text{ to } n)\ p_i\, x_i = p_i\, x_i$

Key Operations

Inner product: $\mathbf{p} \cdot \mathbf{x} = p_i\ x_i$

- **Matrix-vector product:** $\mathbf{P}\ \mathbf{x} = P_{ij}\ x_j$
- **Gradient:** ∇p
- **Divergence:** $\nabla \cdot \mathbf{p}$
- **Time derivative:** dp/dt or $\dot{p}$

Plastic Accounting Notation

- M = Mass of plastic (kg or tonnes)
- S = Stock of plastic in a compartment (kg)
- F = Flow or flux of plastic (kg/year)
- P = Production rate (kg/year)
- C = Consumption rate (kg/year)
- W = Waste generation rate (kg/year)
- R = Recycling rate (kg/year)
- L = Leakage rate to environment (kg/year)
- D = Degradation rate (kg/year)
- TC = Transfer coefficient (dimensionless, $0 \leq TC \leq 1$)
- η = Efficiency (dimensionless, $0 \leq \eta \leq 1$)
- $\lambda(t)$ = Lifetime distribution function ($year^{-1}$)
- k = Rate constant ($year^{-1}$)
- $t_1/_2$ = Half-life (years)
- τ = Mean residence time or lifetime (years)
- E_a = Activation energy (J/mol)
- PF = Plastic footprint (kg)
- MCI = Material Circularity Indicator (dimensionless, $0 \leq MCI \leq 1$)

Subscripts

- vir = Virgin plastic

- rec = Recycled plastic
- in_{use} = In-use stock
- eol = End-of-life
- leak = Leakage
- i, j, k = Indices for compartments, processes, or time steps

Units

Unless otherwise specified: - Mass: kilograms (kg) or metric tonnes (t) - Time: years (year) - Concentration: kg/m^3 - Velocity: m/s - Temperature: Kelvin (K) or Celsius (°C)

PART I: FOUNDATIONS OF PLASTIC ACCOUNTING

Chapter 1: FUNDAMENTALS OF PLASTIC ACCOUNTING

1.1 Introduction

Plastic accounting is the systematic quantification of plastic mass flows and stocks across defined system boundaries. Unlike traditional accounting, which tracks monetary value, plastic accounting tracks physical mass through production, use, waste, recycling, and environmental leakage pathways. The fundamental principle underlying all plastic accounting is the **conservation of mass.**

1.2 The Plastic Mass Balance Equation

Theorem 1.1 (General Mass Balance for Plastic Systems)

For any defined system with boundary Ω, the change in plastic stock S over time interval $[t, t+\Delta t]$ equals the difference between inflows and outflows:

$$\Delta S = \Sigma F_{in} - \Sigma F_{out}$$

or in differential form:

$$dS/dt = \Sigma F_{in} - \Sigma F_{out}$$

where F_{in} represents all inflows (production, imports, recycling) and F_{out} represents all outflows (consumption, exports, waste, leakage).

Proof:

Step 1: Define the system boundary Ω enclosing volume V.

Step 2: Apply conservation of mass. The total mass M within V can only change through transport across the boundary:

$$dM/dt = \iint_{\partial}\Omega \ \rho \ v \cdot n \ dA$$

where ρ is density, v is velocity, n is outward normal, and $\partial\Omega$ is the boundary surface.

Step 3: For discrete flows F_i crossing the boundary:

$$dM/dt = \Sigma_i F_{i,in} - \Sigma_j F_{j,out}$$

Step 4: Since plastic stock S = M for our system:

$$dS/dt = \Sigma F_{in} - \Sigma F_{out}$$

∎

1.3 System Boundaries and Scales

Plastic accounting can be applied at multiple scales:

1. **Global scale:** Entire planet as system boundary
2. **National scale:** Political borders define boundary
3. **Regional/Municipal scale:** Geographic region or city
4. **Corporate scale:** Organizational boundary
5. **Product scale:** Single product or product family
6. **Facility scale:** Single production or waste facility

The choice of system boundary determines which flows are considered internal (transfers) versus external (inflows/outflows).

1.4 Plastic Polymer Types

The seven major polymer categories account for >90% of global plastic production:

1. **Polyethylene (PE):** LDPE, HDPE - packaging, films
2. **Polypropylene (PP):** packaging, automotive
3. **Polyvinyl Chloride (PVC):** construction, pipes
4. **Polystyrene (PS):** packaging, insulation
5. **Polyethylene Terephthalate (PET):** bottles, textiles
6. **Polyurethane (PU):** foams, coatings
7. **Other plastics:** PA, PC, PMMA, etc.

Each polymer has distinct properties affecting degradation, recyclability, and environmental fate.

WORKED EXAMPLES

Example 1: National Plastic Mass Balance

Given: A country has the following annual plastic flows (in million tonnes/year): - Virgin plastic production: P = 15 Mt/yr - Imports: I = 8 Mt/yr - Exports: E = 5 Mt/yr - Waste incineration: W_{inc} = 6 Mt/yr - Recycling: R = 3 Mt/yr - Landfill: W_{lf} = 7 Mt/yr - Environmental leakage: L = 2 Mt/yr

Find: The change in national in-use plastic stock ΔS.

Solution:

Step 1: Identify inflows to the national system.

$F_{in} = P + I = 15 + 8 = 23$ Mt/yr

Step 2: Identify outflows from the national system.

$F_{out} = E + W_{inc} + W_{lf} + L = 5 + 6 + 7 + 2 = 20$ Mt/yr

Step 3: Note that recycling R is an internal flow (waste →production), not an outflow.

Step 4: Apply mass balance equation.

$\Delta S = F_{in} - F_{out} = 23 - 20 = 3$ Mt/yr

Answer: The national in-use plastic stock is increasing by 3 million tonnes per year.

Example 2: Steady-State Condition

Given: A municipal waste management system processes plastic waste at a rate of W = 50,000 tonnes/year. The system has three pathways: - Recycling efficiency: η_R = 0.25 - Incineration fraction: f_{inc} = 0.40 - Landfill fraction: f_{lf} = 0.35

Find: The mass flows through each pathway and verify mass balance closure.

Solution:

Step 1: Calculate recycling flow.

$R = \eta_R \times W = 0.25 \times 50{,}000 = 12{,}500$ tonnes/year

Step 2: Calculate incineration flow.

$W_{inc} = f_{inc} \times W = 0.40 \times 50{,}000 = 20{,}000$ tonnes/year

Step 3: Calculate landfill flow.

$W_{lf} = f_{lf} \times W = 0.35 \times 50{,}000 = 17{,}500$ tonnes/year

Step 4: Verify mass balance (all waste must be accounted for).

$R + W_{inc} + W_{lf} = 12{,}500 + 20{,}000 + 17{,}500 = 50{,}000$ tonnes/year ✓

Step 5: Check fractions sum to 1.

$\eta_R + f_{inc} + f_{lf} = 0.25 + 0.40 + 0.35 = 1.00$ ✓

Answer: Recycling = 12,500 t/yr, Incineration = 20,000 t/yr, Landfill = 17,500 t/yr. Mass balance is closed.

Example 3: Corporate Plastic Footprint

Given: A manufacturing company uses plastic in three categories: - Direct plastic consumption (Scope 1): $M_1 = 500$ tonnes/year - Plastic in purchased components (Scope 2): $M_2 = 1{,}200$ tonnes/year - Plastic in product packaging sold (Scope 3): $M_3 = 800$ tonnes/year

Find: The total corporate plastic footprint PF.

Solution:

Step 1: Define plastic footprint as sum of all scopes.

$PF = M_1 + M_2 + M_3$

Step 2: Substitute values.

```
PF = 500 + 1,200 + 800 = 2,500 tonnes/year
```

Step 3: Calculate scope percentages.

```
Scope 1: 500/2,500 = 20%
Scope 2: 1,200/2,500 = 48%
Scope 3: 800/2,500 = 32%
```

Answer: Total plastic footprint = 2,500 tonnes/year, with Scope 2 (embedded plastic) being the largest contributor at 48%.

Example 4: Recycled Content Calculation

Given: A product contains total plastic mass M_{total} = 2.5 kg, of which: - Virgin plastic: M_{vir} = 1.8 kg - Recycled plastic: M_{rec} = 0.7 kg

Find: The recycled content RC of the product.

Solution:

Step 1: Define recycled content as fraction of total plastic from recycled sources.

$RC = M_{rec} / M_{total}$

Step 2: Substitute values.

```
RC = 0.7 / 2.5 = 0.28
```

Step 3: Express as percentage.

```
RC = 0.28 × 100% = 28%
```

Step 4: Verify mass balance.

$M_{vir} + M_{rec} = 1.8 + 0.7 = 2.5 = M_{total}$ ✓

Answer: The product has 28% recycled content.

Example 5: Leakage Rate Estimation

Given: A coastal city generates plastic waste W = 100,000 tonnes/year. Studies estimate that 3% of waste is mismanaged and enters waterways.

Find: The annual plastic leakage to the marine environment.

Solution:

Step 1: Define leakage as fraction of waste.

$L = f_{leak} \times W$

where $f_{leak} = 0.03$ (leakage fraction).

Step 2: Calculate leakage flow.

```
L = 0.03 × 100,000 = 3,000 tonnes/year
```

Step 3: Express in daily rate.

$L_{daily} = 3{,}000 / 365 = 8.2$ tonnes/day

Answer: The city leaks approximately 3,000 tonnes/year or 8.2 tonnes/day of plastic into the marine environment.

PRACTICE PROBLEMS

Problem 1: A factory produces P = 10,000 tonnes/year of plastic products. It uses virgin plastic (V = 7,500 tonnes/year) and recycled plastic (R_{in} = 2,500 tonnes/year). Manufacturing waste is W_{mfg} = 800 tonnes/year, of which 60% is recycled internally. Calculate: (a) the mass balance for the factory, (b) the recycled content of products, and (c) the net virgin plastic consumption.

Problem 2: A national plastic system has production P = 50 Mt/yr, imports I = 20 Mt/yr, exports E = 15 Mt/yr, and waste generation W = 45 Mt/yr. If the in-use stock is growing at 10 Mt/yr, calculate the unaccounted losses (leakage + degradation).

Problem 3: A product contains 5 kg of plastic. After 10 years of use, it is collected for recycling. The recycling process has 85% efficiency (15% lost as residue). Calculate the mass of recycled plastic output and verify mass conservation.

Problem 4: A municipality collects 80,000 tonnes/year of plastic waste. The waste management system has: recycling rate RR = 0.30, incineration rate IR = 0.50, and landfill rate LR = 0.20. Calculate the mass flow through each pathway and verify that RR + IR + LR = 1.

Problem 5: A global plastic production model estimates annual production growth rate g = 3.5% per year. If current production is P_0 = 400 Mt/yr, derive an expression for production P(t) after t years and calculate production in 2050 (t = 25 years from 2025).

Chapter 2: PLASTIC MASS BALANCE AND FLOW ANALYSIS

2.1 Introduction to Material Flow Analysis (MFA)

Material Flow Analysis is a systematic assessment of flows and stocks of materials within a defined system in space and time. For plastic accounting, MFA provides the quantitative foundation for tracking plastic through the economy and environment.

2.2 The MFA Framework

Definition 2.1 (Material Flow Analysis System)

An MFA system consists of: 1. **Processes** (P): Activities that transform, transport, or store materials 2. **Flows** (F): Material transfers between processes 3. **Stocks** (S): Material accumulations within processes 4. **System boundary** (Ω): Spatial and temporal limits

2.3 The Technology Matrix

Theorem 2.1 (Technology Matrix Formulation)

For a system with n processes, the technology matrix **A** $\in \mathbb{R}^{n \times n}$ describes input-output relationships:

A_{ij} = (mass of plastic from process i to process j) / (total output of process i)

The steady-state flow vector **f** satisfies:

```
(**I** - **A**ᵀ) **f** = **d**
```

where **I** is the identity matrix and **d** is the final demand vector.

Proof:

Step 1: For each process j, mass balance requires:

$f_j = \Sigma_i A_{ij} f_i + d_j$

Step 2: In matrix form:

```
**f** = **A**ᵀ **f** + **d**
```

Step 3: Rearrange:

```
**f** - **A**ᵀ **f** = **d**
(**I** - **A**ᵀ) **f** = **d**
```

Step 4: If (**I** - $\mathbf{A}^T$) is invertible (Leontief condition):

```
**f** = (**I** - **A**ᵀ)⁻¹ **d**
```

∎

2.4 Dynamic Mass Balance

For time-dependent systems:

$dS_i/dt = \Sigma_j F_{ji} - \Sigma_k F_{ik}$

where S_i is stock in process i, F_{ji} is flow from j to i, and F_{ik} is flow from i to k.

WORKED EXAMPLES

Example 1: Three-Process MFA System

Given: A simplified plastic system has three processes: 1. Production (P): Produces 100 tonnes/year 2. Use (U): Consumes products 3. Waste Management (W): Processes end-of-life plastic

Transfer coefficients: - Production →Use: TC_{PU} = 0.95 - Production →Waste (manufacturing scrap): TC_{PW} = 0.05 - Use →Waste: TC_{UW} = 1.00 (all products eventually become waste) - Waste → Production (recycling): TC_{WP} = 0.30 - Waste →Environment (landfill + leakage): TC_{WE} = 0.70

Find: All flow values and verify mass balance for each process.

Solution:

Step 1: Calculate flows from Production.

$F_{PU} = TC_{PU} \times F_P = 0.95 \times 100 = 95$ tonnes/year

$F_{PW} = TC_{PW} \times F_P = 0.05 \times 100 = 5$ tonnes/year

Step 2: Assume steady state for Use process (stock not changing).

$F_{in,U} = F_{out,U}$

$F_{PU} = F_{UW}$

$F_{UW} = 95$ tonnes/year

Step 3: Calculate total waste input.

$F_{in,W} = F_{PW} + F_{UW} = 5 + 95 = 100$ tonnes/year

Step 4: Calculate flows from Waste.

$F_{WP} = TC_{WP} \times F_{in,W} = 0.30 \times 100 = 30$ tonnes/year (recycling)

$F_{WE} = TC_{WE} \times F_{in,W} = 0.70 \times 100 = 70$ tonnes/year (disposal)

Step 5: Verify mass balance for Production process.

Inflow: $F_{WP} = 30$ tonnes/year (recycled plastic)

Virgin input: $V = 100 - 30 = 70$ tonnes/year

Outflow: $F_{PU} + F_{PW} = 95 + 5 = 100$ tonnes/year

Balance: $70 + 30 = 100$ ✓

Answer: Virgin plastic input = 70 t/yr, Recycling = 30 t/yr, Environmental disposal = 70 t/yr.

Example 2: Dynamic Stock Accumulation

Given: A country's in-use plastic stock $S(t)$ grows according to: - Annual consumption: $C = 20$ Mt/year (constant) - Waste generation: $W(t) = 0.05 \times S(t)$ (5% of stock becomes waste each year)

Initial stock: $S(0) = 200$ Mt

Find: The stock $S(t)$ after 10 years.

Solution:

Step 1: Write the differential equation for stock.

```
dS/dt = C - W(t) = C - 0.05S
```

Step 2: Substitute values.

```
dS/dt = 20 - 0.05S
```

Step 3: This is a first-order linear ODE. Rearrange:

```
dS/dt + 0.05S = 20
```

Step 4: Solve using integrating factor μ = e^(0.05t).

```
d/dt[S e^(0.05t)] = 20 e^(0.05t)
```

Step 5: Integrate both sides.

```
S e^(0.05t) = (20/0.05) e^(0.05t) + K
S e^(0.05t) = 400 e^(0.05t) + K
S(t) = 400 + K e^(-0.05t)
```

Step 6: Apply initial condition S(0) = 200.

```
200 = 400 + K
K = -200
```

Step 7: Final solution.

```
S(t) = 400 - 200 e^(-0.05t)
```

Step 8: Calculate S(10).

```
S(10) = 400 - 200 e^(-0.5) = 400 - 200(0.6065) = 400 - 121.3 = 278.7 Mt
```

Answer: After 10 years, the in-use stock will be approximately 279 million tonnes.

Example 3: Recycling Loop Analysis

Given: A closed-loop recycling system has: - Virgin plastic input: V = 1000 kg/year - Recycling efficiency: η = 0.80 (20% loss in recycling) - Product lifetime: τ = 5 years (average)

Assume steady state.

Find: The total plastic in circulation (including recycled material).

Solution:

Step 1: In steady state, waste generation W equals consumption C.

$$W = C$$

Step 2: Consumption equals virgin input plus recycled input.

$$C = V + R$$

where R is recycled plastic.

Step 3: Recycled plastic equals waste times efficiency.

$$R = \eta \times W = \eta \times C$$

Step 4: Substitute into consumption equation.

$$C = V + \eta C$$
$$C - \eta C = V$$
$$C(1 - \eta) = V$$
$$C = V / (1 - \eta)$$

Step 5: Calculate consumption.

$$C = 1000 / (1 - 0.80) = 1000 / 0.20 = 5000 \text{ kg/year}$$

Step 6: Calculate recycling flow.

$$R = \eta C = 0.80 \times 5000 = 4000 \text{ kg/year}$$

Step 7: Calculate in-use stock (for exponential lifetime distribution).

$$S = C \times \tau = 5000 \times 5 = 25{,}000 \text{ kg}$$

Answer: Total consumption = 5000 kg/yr (1000 virgin + 4000 recycled), In-use stock = 25,000 kg.

Example 4: Multi-Compartment System

Given: A plastic flow system has four compartments: - Production (P): 100 units/year output - Manufacturing (M): Receives from P - Retail (R): Receives from M - Waste (W): Receives from R

Transfer efficiencies: - P →M: 98% (2% production loss) - M →R: 95% (5% manufacturing waste) - R →W: 100% (all products eventually discarded)

Find: Flow into each compartment.

Solution:

Step 1: Flow from Production to Manufacturing.

$F_{PM} = 0.98 \times 100 = 98$ units/year

Step 2: Production loss.

$F_{P,loss} = 0.02 \times 100 = 2$ units/year

Step 3: Flow from Manufacturing to Retail.

$F_{MR} = 0.95 \times 98 = 93.1$ units/year

Step 4: Manufacturing waste.

$F_{M,waste} = 0.05 \times 98 = 4.9$ units/year

Step 5: Flow from Retail to Waste.

$F_{RW} = 1.00 \times 93.1 = 93.1$ units/year

Step 6: Total waste.

$W_{total} = F_{P,loss} + F_{M,waste} + F_{RW} = 2 + 4.9 + 93.1 = 100$ units/year

Step 7: Verify overall mass balance.

```
Input (Production) = Output (Waste) = 100 units/year ✓
```

Answer: Retail receives 93.1 units/year, total waste = 100 units/year (matching production).

Example 5: Import-Export Balance

Given: A country has: - Domestic production: P = 30 Mt/year - Imports: I = 15 Mt/year - Exports: E = 10 Mt/year - Domestic consumption: C = ?

Assume steady state (no stock change).

Find: Domestic consumption C.

Solution:

Step 1: Apply mass balance for the national system.

```
Inflows = Outflows
P + I = C + E
```

Step 2: Solve for consumption.

```
C = P + I − E
```

Step 3: Substitute values.

```
C = 30 + 15 − 10 = 35 Mt/year
```

Step 4: Calculate trade balance.

```
Net imports = I − E = 15 − 10 = 5 Mt/year
```

Step 5: Verify: Consumption = Domestic production + Net imports.

```
C = 30 + 5 = 35 Mt/year ✓
```

Answer: Domestic consumption = 35 million tonnes/year.

PRACTICE PROBLEMS

Problem 1: A plastic production facility has three output streams: products (90%), manufacturing waste sent to recycling (7%), and losses (3%). If total production is 50,000 tonnes/year, calculate the mass flow in each stream and verify mass conservation.

Problem 2: Derive the steady-state solution for a two-process system (Production and Use) with recycling. If production $P = 100$ units/year, recycling efficiency $\eta = 0.40$, and product lifetime $\tau = 8$ years, calculate the in-use stock S.

Problem 3: A dynamic MFA model has $dS/dt = 15 - 0.03S$, where S is stock in million tonnes and t is in years. If $S(0) = 100$ Mt, solve for $S(t)$ and find the equilibrium stock as $t \to \infty$.

Problem 4: Create a technology matrix **A** for a three-process system (Production, Use, Waste) with the following transfer coefficients: Production→Use (0.95), Production→Waste (0.05), Use→Waste (1.0), Waste→Production (0.25). Calculate the Leontief inverse $(\mathbf{I} - \mathbf{A}^T)^{-1}$.

Problem 5: A country imports 20% of its plastic consumption and exports 15% of its production. If consumption $C = 40$ Mt/year, calculate domestic production P, imports I, and exports E.

Chapter 3: PLASTIC LIFECYCLE DYNAMICS

3.1 Introduction

Plastic products have finite lifetimes, after which they enter the waste stream. Understanding the distribution of product lifetimes is essential for predicting waste generation and modeling plastic stocks and flows dynamically.

3.2 Lifetime Distributions

Definition 3.1 (Product Lifetime Distribution)

The lifetime of a plastic product is a random variable T with probability density function (PDF) $\lambda(t)$, where: - $\lambda(t)dt$ = probability that a product fails in interval [t, t+dt] - $\int_0^\infty \lambda(t)\, dt = 1$ (normalization)

The survival function S(t) gives the probability a product survives beyond time t:

$$S(t) = \int_t^\infty \lambda(\tau)\, d\tau = 1 - \int_0^t \lambda(\tau)\, d\tau$$

3.3 Common Lifetime Distributions

Exponential Distribution

$$\lambda(t) = (1/\tau) \exp(-t/\tau)$$

Mean lifetime: τ

Memoryless property: $P(T > s+t \mid T > s) = P(T > t)$

Weibull Distribution

$$\lambda(t) = (k/\tau)(t/\tau)^{(k-1)} \exp(-(t/\tau)^k)$$

Shape parameter k: - $k < 1$: decreasing failure rate (infant mortality) - $k = 1$: constant failure rate (exponential) - $k > 1$: increasing failure rate (wear-out)

Normal Distribution (truncated at t=0)

$$\lambda(t) = (1/(\sigma\sqrt{2\pi})) \exp(-(t-\mu)^2/(2\sigma^2)) \quad \text{for } t \geq 0$$

Mean lifetime: μ

Standard deviation: σ

3.4 Delay Differential Equations

Theorem 3.1 (Waste Generation from Consumption History)

If consumption at time t is C(t) and products have lifetime distribution $\lambda(\tau)$, then waste generation W(t) is:

$$W(t) = \int_0^\infty C(t-\tau)\ \lambda(\tau)\ d\tau$$

This is a convolution integral relating current waste to past consumption.

Proof:

Step 1: Consider products consumed at time $t-\tau$ (τ years ago).

Step 2: The probability these products fail at time t is $\lambda(\tau)d\tau$.

Step 3: The mass of products consumed at $t-\tau$ is $C(t-\tau)$.

Step 4: The waste generated at time t from products consumed at $t-\tau$ is:

$$dW = C(t-\tau)\ \lambda(\tau)\ d\tau$$

Step 5: Integrate over all past consumption:

$$W(t) = \int_0^\infty C(t-\tau)\ \lambda(\tau)\ d\tau$$

∎

WORKED EXAMPLES

Example 1: Exponential Lifetime - Mean and Variance

Given: Plastic packaging has exponential lifetime distribution with mean $\tau = 0.5$ years.

Find: (a) The PDF $\lambda(t)$, (b) the probability a package survives beyond 1 year, (c) the variance.

Solution:

Step 1: Write the PDF for exponential distribution.

```
λ(t) = (1/τ) exp(-t/τ) = (1/0.5) exp(-t/0.5) = 2 exp(-2t)
```

Step 2: Calculate survival probability S(1).

```
S(t) = exp(-t/τ)
S(1) = exp(-1/0.5) = exp(-2) = 0.135
```

Step 3: Calculate variance. For exponential distribution: $Var(T) = \tau^2$

```
Var(T) = (0.5)² = 0.25 years²
```

Answer: (a) $\lambda(t) = 2\exp(-2t)$, (b) $P(T > 1) = 13.5\%$, (c) Variance = 0.25 years^2.

Example 2: Weibull Distribution - Shape Parameter Effect

Given: Two plastic products have Weibull lifetime distributions with scale $\tau = 10$ years: - Product A: $k = 0.5$ (packaging) - Product B: $k = 3.0$ (durable good)

Find: The mean lifetime for each product.

Solution:

Step 1: Recall mean of Weibull distribution.

```
E[T] = τ Γ(1 + 1/k)
```

where Γ is the gamma function.

Step 2: Calculate for Product A ($k = 0.5$).

$E[T_A] = 10 \times \Gamma(1 + 1/0.5) = 10 \times \Gamma(3) = 10 \times 2 = 20$ years

Step 3: Calculate for Product B ($k = 3.0$).

$E[T_B] = 10 \times \Gamma(1 + 1/3) = 10 \times \Gamma(1.333) \approx 10 \times 0.893 = 8.93$ years

Step 4: Interpret results. - Product A ($k < 1$): High early failure rate, but survivors last long → mean > scale - Product B ($k > 1$): Increasing failure rate with age →mean < scale

Answer: Product A mean = 20 years, Product B mean ≈ 8.93 years.

Example 3: Waste Generation from Constant Consumption

Given: Consumption is constant: $C(t) = C_0 = 100$ tonnes/year for all $t \geq 0$. Lifetime distribution is exponential: $\lambda(\tau) = (1/\tau) \exp(-\tau/\tau)$ with $\tau = 5$ years.

Find: The waste generation rate $W(t)$ for $t \to \infty$ (steady state).

Solution:

Step 1: Apply the waste generation formula.

$$W(t) = \int_0^\infty C(t-\tau)\ \lambda(\tau)\ d\tau$$

Step 2: Since $C(t) = C_0$ (constant):

$$W(t) = C_0 \int_0^t \lambda(\tau)\ d\tau$$

(Upper limit is t because $C(t-\tau) = 0$ for $\tau > t$ if consumption started at $t=0$)

Step 3: For large t (steady state), $t \to \infty$:

$$W(\infty) = C_0 \int_0^\infty \lambda(\tau)\ d\tau = C_0 \times 1 = C_0$$

Step 4: Substitute value.

$$W(\infty) = 100 \text{ tonnes/year}$$

Step 5: Interpret: In steady state, waste generation equals consumption (mass balance).

Answer: Steady-state waste generation = 100 tonnes/year (equals consumption).

Example 4: Stock Calculation with Exponential Lifetime

Given: Consumption C = 50 tonnes/year (constant). Exponential lifetime with mean $\tau = 8$ years.

Find: The in-use stock S in steady state.

Solution:

Step 1: For exponential lifetime, the stock-consumption relationship is:

$$S = C \times \tau$$

This can be derived from the integral:

$$S = \int_0^\infty C \times S(t)\, dt$$

where $S(t) = \exp(-t/\tau)$ is the survival function.

Step 2: Evaluate the integral.

$$S = C \int_0^\infty \exp(-t/\tau)\, dt = C \times \tau$$

Step 3: Substitute values.

$$S = 50 \times 8 = 400 \text{ tonnes}$$

Answer: In-use stock = 400 tonnes.

Example 5: Normal Lifetime Distribution

Given: A durable plastic product has normally distributed lifetime with: - Mean: $\mu = 15$ years - Standard deviation: $\sigma = 3$ years

Find: The probability a product fails before 10 years.

Solution:

Step 1: Standardize the variable.

$$Z = (T - \mu) / \sigma = (10 - 15) / 3 = -5/3 = -1.667$$

Step 2: Find cumulative probability.

$P(T < 10) = P(Z < -1.667) = \Phi(-1.667)$

where Φ is the standard normal CDF.

Step 3: Use standard normal table or calculator.

$\Phi(-1.667) \approx 0.0478$

Answer: Approximately 4.78% of products fail before 10 years.

PRACTICE PROBLEMS

Problem 1: A plastic product has Weibull lifetime distribution with $\tau = 12$ years and $k = 2.5$. Calculate: (a) the mean lifetime, (b) the median lifetime (50th percentile), and (c) the 90th percentile lifetime.

Problem 2: Consumption grows exponentially: $C(t) = C_0 \exp(gt)$ with $C_0 = 100$ tonnes/year and $g = 0.03$/year. Products have exponential lifetime with $\tau = 6$ years. Derive an expression for waste generation $W(t)$ using the convolution integral.

Problem 3: Prove that for exponential lifetime distribution, the in-use stock $S = C \times \tau$, where C is constant consumption and τ is mean lifetime.

Problem 4: A product has triangular lifetime distribution: $\lambda(t) = 2t/T^2$ for $0 \leq t \leq T$, and $\lambda(t) = 0$ otherwise. Calculate the mean lifetime and variance.

Problem 5: Compare two scenarios: (A) Exponential lifetime with $\tau = 10$ years, (B) Constant lifetime (all products fail at exactly $t = 10$ years). If consumption is constant at $C = 100$ tonnes/year, calculate the steady-state in-use stock for each scenario.

Chapter 4: DEGRADATION AND FRAGMENTATION KINETICS

4.1 Introduction

Unlike organic materials that biodegrade relatively quickly, conventional plastics persist in the environment for decades to centuries. Degradation processes transform plastics into smaller fragments (microplastics and nanoplastics) rather than complete mineralization.

4.2 First-Order Degradation Kinetics

Theorem 4.1 (First-Order Decay Law)

If plastic degrades at a rate proportional to its mass, then:

$$dM/dt = -k\,M$$

where k is the degradation rate constant (year^{-1}).

Solution:

$$M(t) = M_0 \exp(-kt)$$

Half-life:

$$t_{1/2} = \ln(2) / k \approx 0.693 / k$$

Proof:

Step 1: Separate variables.

$$dM/M = -k\,dt$$

Step 2: Integrate both sides.

$$\int dM/M = -k \int dt$$
$$\ln M = -kt + C$$

Step 3: Solve for M.

$$M = \exp(-kt + C) = \exp(C)\exp(-kt) = M_0 \exp(-kt)$$

where $M_0 = \exp(C)$ is the initial mass.

Step 4: Find half-life by setting $M(t_1/_2) = M_0/2$.

```
M₀/2 = M₀ exp(-k t₁/₂)
1/2 = exp(-k t₁/₂)
ln(1/2) = -k t₁/₂
t₁/₂ = -ln(1/2) / k = ln(2) / k
```

∎

4.3 Temperature Dependence - Arrhenius Equation

Theorem 4.2 (Arrhenius Temperature Dependence)

The degradation rate constant k depends on temperature T according to:

$k(T) = A \exp(-E_a / (R\ T))$

where: - A = pre-exponential factor ($year^{-1}$) - E_a = activation energy (J/mol) - R = gas constant = 8.314 J/(mol·K) - T = absolute temperature (K)

Proof: Derived from transition state theory in chemical kinetics (beyond scope).

4.4 Fragmentation Cascade

Plastic degradation produces a cascade of smaller fragments:

```
Macroplastic →Microplastic (< 5 mm) →Nanoplastic (< 1 μm) →Molecular
degradation
```

Each stage has its own kinetics and environmental implications.

WORKED EXAMPLES

Example 1: Half-Life Calculation

Given: Polyethylene (PE) in marine environment has degradation rate constant $k = 0.01\ year^{-1}$.

Find: The half-life $t_1/_2$.

Solution:

Step 1: Apply half-life formula.

$$t_{1/2} = \ln(2) / k$$

Step 2: Substitute value.

$$t_{1/2} = 0.693 / 0.01 = 69.3 \text{ years}$$

Answer: Half-life of PE in marine environment ≈ 69 years.

Example 2: Mass Remaining After Time t

Given: Initial plastic mass: $M_0 = 1000$ kg Degradation rate: $k = 0.015$ year^{-1} Time elapsed: $t = 50$ years

Find: Mass remaining M(50).

Solution:

Step 1: Apply first-order decay equation.

$$M(t) = M_0 \exp(-kt)$$

Step 2: Substitute values.

$$M(50) = 1000 \times \exp(-0.015 \times 50)$$
$$M(50) = 1000 \times \exp(-0.75)$$
$$M(50) = 1000 \times 0.472$$
$$M(50) = 472 \text{ kg}$$

Step 3: Calculate fraction degraded.

$$\text{Fraction degraded} = (M_0 - M(50)) / M_0 = (1000 - 472) / 1000 = 0.528 = 52.8\%$$

Answer: After 50 years, 472 kg remains (52.8% has degraded).

Example 3: Arrhenius Temperature Effect

Given: At T_1 = 298 K (25°C), degradation rate k_1 = 0.01 $year^{-1}$. Activation energy E_a = 80 kJ/mol = 80,000 J/mol.

Find: Degradation rate k_2 at T_2 = 308 K (35°C).

Solution:

Step 1: Use Arrhenius equation ratio.

$k_2 / k_1 = \exp(-E_a/R \times (1/T_2 - 1/T_1))$

Step 2: Calculate temperature term.

$1/T_2 - 1/T_1 = 1/308 - 1/298 = 0.003247 - 0.003356 = -0.000109\ K^{-1}$

Step 3: Calculate exponent.

$-E_a/R \times (1/T_2 - 1/T_1) = -(80000/8.314) \times (-0.000109)$
$= 9620 \times 0.000109 = 1.048$

Step 4: Calculate rate ratio.

$k_2 / k_1 = \exp(1.048) = 2.85$

Step 5: Calculate k_2.

$k_2 = 2.85 \times k_1 = 2.85 \times 0.01 = 0.0285\ year^{-1}$

Answer: At 35°C, degradation rate increases to 0.0285 $year^{-1}$ (2.85× faster than at 25°C).

Example 4: Multi-Stage Degradation

Given: Plastic degrades in two stages: - Stage 1 (macroplastic → microplastic): k_1 = 0.02 $year^{-1}$ - Stage 2 (microplastic → nanoplastic): k_2 = 0.05 $year^{-1}$

Initial macroplastic mass: M_0 = 100 kg

Find: Mass of microplastic $M_{micro}(t)$ as a function of time.

Solution:

Step 1: Write differential equations for two-stage system.

$dM_{macro}/dt = -k_1 M_{macro}$

$dM_{micro}/dt = k_1 M_{macro} - k_2 M_{micro}$

Step 2: Solve first equation.

$M_{macro}(t) = M_0 \exp(-k_1 t)$

Step 3: Substitute into second equation.

$dM_{micro}/dt + k_2 M_{micro} = k_1 M_0 \exp(-k_1 t)$

Step 4: This is a first-order linear ODE. Solution (assuming $M_{micro}(0) = 0$):

$M_{micro}(t) = (k_1 M_0)/(k_2 - k_1) \times [\exp(-k_1 t) - \exp(-k_2 t)]$

Step 5: Substitute values.

$M_{micro}(t) = (0.02 \times 100)/(0.05 - 0.02) \times [\exp(-0.02t) - \exp(-0.05t)]$

$M_{micro}(t) = 2/0.03 \times [\exp(-0.02t) - \exp(-0.05t)]$

$M_{micro}(t) = 66.67 \times [\exp(-0.02t) - \exp(-0.05t)]$

Answer: $M_{micro}(t) = 66.67[\exp(-0.02t) - \exp(-0.05t)]$ kg

---

Example 5: Time to 90% Degradation

Given: Polystyrene (PS) has degradation rate $k = 0.012$ year^{-1}.

Find: Time required for 90% degradation.

Solution:

Step 1: Set up equation for 10% remaining.

```
M(t) / M₀ = 0.10
```

Step 2: Use decay equation.

```
exp(-kt) = 0.10
```

Step 3: Take natural logarithm.

```
-kt = ln(0.10) = -2.303
```

Step 4: Solve for t.

```
t = 2.303 / k = 2.303 / 0.012 = 192 years
```

Answer: 90% degradation requires approximately 192 years.

PRACTICE PROBLEMS

Problem 1: A plastic item has half-life $t_1/_2$ = 80 years. Calculate the degradation rate constant k and the time required for 99% degradation.

Problem 2: Derive the relationship between half-life $t_1/_2$ and mean lifetime τ for first-order degradation kinetics. Show that τ = $t_1/_2$ / ln(2).

Problem 3: At 20°C, k = 0.008 year^{-1}. If activation energy E_a = 75 kJ/mol, calculate the degradation rate at 0°C and at 40°C.

Problem 4: For the two-stage degradation model in Example 4, calculate the time at which microplastic mass $M_{micro}(t)$ reaches its maximum value. (Hint: Set dM_{micro}/dt = 0)

Problem 5: A plastic debris field has initial mass M_0 = 10,000 tonnes. If degradation follows first-order kinetics with k = 0.01 year^{-1}, calculate: (a) mass remaining after 100 years, (b) total mass degraded in first 100 years, (c) degradation rate (dM/dt) at t = 100 years.

PART II: MATHEMATICAL METHODS

Chapter 5: LINEAR ALGEBRA FOR PLASTIC SYSTEMS ANALYSIS

5.1 Introduction

Linear algebra provides the mathematical framework for analyzing complex plastic flow networks with multiple processes and interconnections. Matrix methods enable systematic solution of mass balance equations and analysis of system properties.

5.2 Matrix Representation of Flow Networks

Definition 5.1 (Flow Matrix)

For a system with n processes, the flow matrix $\mathbf{F} \in \mathbb{R}^{n \times n}$ has elements:

F_{ij} = flow from process i to process j (kg/year)

The total outflow from process i is:

$F_{i,out} = \Sigma_j F_{ij}$

5.3 The Leontief Inverse

Theorem 5.1 (Leontief Input-Output Model)

For technology matrix A where $A_{ij} = F_{ij} / F_{j,out}$, the total output x required to meet final demand d is:

x = (**I** - **A**)⁻¹ **d**

The matrix $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$ is called the Leontief inverse.

Proof:

Step 1: Each process j requires inputs from all processes i:

$x_j = \Sigma_i A_{ij} x_i + d_j$

Step 2: In matrix form:

```
**x** = **A** **x** + **d**
```

Step 3: Rearrange:

```
**x** - **A** **x** = **d**
(**I** - **A**) **x** = **d**
```

Step 4: If (**I** - **A**) is non-singular:

```
**x** = (**I** - **A**)^-1 **d** = **L** **d**
```

∎

5.4 Eigenvalue Analysis

Theorem 5.2 (Stability Condition)

A plastic flow system is stable if all eigenvalues λ_i of the technology matrix A satisfy $|\lambda_i| < 1$.

WORKED EXAMPLES

Example 1: 2×2 System - Matrix Inversion

Given: Two-process system (Production P, Use U) with flows: - Virgin input to P: 100 kg/yr - P → U: 95 kg/yr - U → P (recycling): 30 kg/yr - U → Waste: 65 kg/yr

Find: The technology matrix **A** and Leontief inverse **L**.

Solution:

Step 1: Calculate total outputs.

$F_{P,out}$ = 95 kg/yr

$F_{U,out}$ = 30 + 65 = 95 kg/yr

Step 2: Construct technology matrix.

A_{PP} = 0 (no P → P flow)

A_{PU} = 30/95 = 0.316

A_{UP} = 95/95 = 1.0

A_{UU} = 0 (no U → U flow)

A = [0 0.316]

[1.0 0]

Step 3: Calculate **I** - **A**.

```
**I** - **A** = [ 1    -0.316 ]
                [-1      1    ]
```

Step 4: Calculate determinant.

```
det(**I** - **A**) = 1×1 - (-0.316)×(-1) = 1 - 0.316 = 0.684
```

Step 5: Calculate inverse using formula for 2×2 matrix.

```
(**I** - **A**)⁻¹ = (1/0.684) [  1      0.316 ]
                              [  1      1     ]


**L** = [ 1.462   0.462 ]
        [ 1.462   1.462 ]
```

Answer: Leontief inverse **L** shows that to produce 1 unit of final demand from Production requires 1.462 units of total production activity.

Example 2: Matrix-Vector Multiplication

Given: Technology matrix from Example 1 and final demand vector:

```
**d** = [ 70 ]   (70 kg/yr to external customers)
        [ 0  ]   (no direct demand from Use)
```

Find: Total output vector **x**.

Solution:

Step 1: Apply Leontief model.

```
**x** = **L** **d**
```

Step 2: Perform matrix-vector multiplication.

x = [1.462   0.462] [70]
        [1.462   1.462] [0]

$x_P = 1.462 \times 70 + 0.462 \times 0 = 102.3$ kg/yr

$x_U = 1.462 \times 70 + 1.462 \times 0 = 102.3$ kg/yr

Step 3: Verify with virgin input. Virgin = 100 kg/yr, Recycling = 30 kg/yr Total P output = 100 + 30 - 30 (internal) = 100 kg/yr ≈ 102.3 (close, difference due to rounding)

Answer: Total production output = 102.3 kg/yr, Total use = 102.3 kg/yr.

Example 3: Eigenvalue Calculation

Given: Technology matrix:

```
**A** = [ 0.2    0.3 ]
        [ 0.4    0.1 ]
```

Find: Eigenvalues and assess stability.

Solution:

Step 1: Write characteristic equation.

```
det(**A** - λ**I**) = 0
det([ 0.2-λ    0.3   ]) = 0
   ([ 0.4      0.1-λ ])
```

Step 2: Expand determinant.

$(0.2-\lambda)(0.1-\lambda) - (0.3)(0.4) = 0$

$0.02 - 0.2\lambda - 0.1\lambda + \lambda^2 - 0.12 = 0$

$\lambda^2 - 0.3\lambda - 0.10 = 0$

Step 3: Solve quadratic equation.

```
λ = (0.3 ± √(0.09 + 0.40)) / 2
λ = (0.3 ± √0.49) / 2
λ = (0.3 ± 0.7) / 2
```

Step 4: Calculate eigenvalues.

```
λ1 = (0.3 + 0.7) / 2 = 0.5
λ2 = (0.3 - 0.7) / 2 = -0.2
```

Step 5: Check stability condition.

```
|λ1| = 0.5 < 1 √
|λ2| = 0.2 < 1 √
```

Answer: Eigenvalues are $\lambda_1 = 0.5$, $\lambda_2 = -0.2$. System is stable since $|\lambda_i| < 1$ for all i.

Example 4: Three-Process System

Given: Three processes: Production (P), Manufacturing (M), Retail (R). Technology matrix:

```
**A** = [ 0     0     0.2  ]  (R → P recycling)
        [ 0.9   0     0    ]  (P → M)
        [ 0     0.95  0    ]  (M → R)
```

Final demand: **d** = $[0, 0, 100]^T$ (100 units to consumers from R)

Find: Total output **x** for each process.

Solution:

Step 1: Calculate **I** - **A**.

```
**I** - **A** = [ 1     0     -0.2  ]
                [-0.9   1      0    ]
                [ 0    -0.95   1    ]
```

Step 2: Calculate inverse (using calculator or software).

```
(**I** - **A**)⁻¹ ≈ [ 1.198   0.228   0.240 ]
                     [ 1.078   1.205   0.216 ]
                     [ 1.024   1.145   1.205 ]
```

Step 3: Calculate output vector.

```
**x** = **L** **d** = [ 1.198    0.228    0.240 ] [ 0   ]
                      [ 1.078    1.205    0.216 ] [ 0   ]
                      [ 1.024    1.145    1.205 ] [ 100 ]
```

$x_P = 0.240 \times 100 = 24.0$

$x_M = 0.216 \times 100 = 21.6$

$x_R = 1.205 \times 100 = 120.5$

Answer: Production = 24.0 units, Manufacturing = 21.6 units, Retail = 120.5 units.

Example 5: Rank and Linear Independence

Given: Flow matrix for a system:

```
**F** = [ 0     50    0  ]
        [ 0     0     45 ]
        [ 10    0     0  ]
```

Find: The rank of **F** and interpret.

Solution:

Step 1: Perform row reduction to echelon form.

```
[ 0     50    0  ]      [ 10    0     0  ]   (swap R1      R3)
[ 0     0     45 ]  ▯   [ 0     50    0  ]   (swap R2      R3)
[ 10    0     0  ]      [ 0     0     45 ]
```

Step 2: The matrix is already in echelon form with 3 non-zero rows.

Step 3: Count pivot positions.

```
rank(**F**) = 3
```

Step 4: Interpret: All three processes are linearly independent; the system has full rank.

Answer: rank(**F**) = 3, indicating all processes are essential (none can be expressed as linear combination of others).

PRACTICE PROBLEMS

Problem 1: For a 2×2 technology matrix **A** = [[0.3, 0.2], [0.4, 0.1]], calculate the Leontief inverse **L** = (**I** - **A**)$^{-1}$ by hand.

Problem 2: Prove that if A is a technology matrix with all elements 0 ≤ A_{ij} < 1 and row sums < 1, then (I - A) is invertible.

Problem 3: A four-process plastic system has technology matrix A. If the largest eigenvalue is λ_{max} = 0.95, what does this imply about the system's stability and convergence rate?

Problem 4: Calculate the determinant of the matrix **I** - **A** where **A** = [[0.2, 0.3, 0.1], [0.3, 0.1, 0.2], [0.1, 0.2, 0.3]]. Is the matrix invertible?

Problem 5: For a recycling system with technology matrix **A** = [[0, 0.3], [0.8, 0]], find the eigenvalues and eigenvectors. Interpret the physical meaning of the dominant eigenvector.

Chapter 6: PROBABILITY AND UNCERTAINTY QUANTIFICATION

6.1 Introduction

Plastic accounting involves significant uncertainties from measurement errors, sampling variability, and model assumptions. Probability theory provides the framework for quantifying and propagating these uncertainties.

6.2 Random Variables and Distributions

Definition 6.1 (Random Variable)

A random variable X is a function mapping outcomes to real numbers. For plastic accounting: - Discrete: $X \in \{x_1, x_2, \ldots\}$ (e.g., number of products) - Continuous: $X \in \mathbb{R}$ (e.g., plastic mass)

Definition 6.2 (Probability Density Function)

For continuous X, the PDF f(x) satisfies:

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

6.3 Common Distributions for MFA

Normal Distribution

```
f(x) = (1/(σ√(2π))) exp(-(x-μ)²/(2σ²))
```

Parameters: mean μ, standard deviation σ

Lognormal Distribution

```
f(x) = (1/(xσ√(2π))) exp(-(ln x - μ)²/(2σ²))
```

For positive quantities with multiplicative errors.

Uniform Distribution

```
f(x) = 1/(b-a)   for a ≤ x ≤ b
```

Maximum entropy distribution for bounded support.

Triangular Distribution

```
f(x) = { 2(x-a)/((b-a)(c-a))   for a ≤ x ≤ c
       { 2(b-x)/((b-a)(b-c))   for c < x ≤ b
```

Parameters: min a, mode c, max b.

6.4 Uncertainty Propagation

Theorem 6.1 (Linear Uncertainty Propagation)

For function $Y = f(X_1, X_2, \ldots, X_n)$ with independent inputs:

$\sigma_Y^2 \approx \Sigma i\ (\partial f/\partial Xi)^2\ \sigma X_i^2$

This is first-order Taylor series approximation.

WORKED EXAMPLES

Example 1: Normal Distribution Parameters

Given: Plastic waste generation is normally distributed with mean μ = 50,000 tonnes/year and standard deviation σ = 5,000 tonnes/year.

Find: (a) Probability that waste generation exceeds 60,000 tonnes/year, (b) 95% confidence interval.

Solution:

Step 1: Standardize for part (a).

```
Z = (X - μ) / σ = (60000 - 50000) / 5000 = 2.0
```

Step 2: Find probability.

```
P(X > 60000) = P(Z > 2.0) = 1 - Φ(2.0) = 1 - 0.9772 = 0.0228
```

Step 3: Calculate 95% CI ($z = 1.96$ for 95%).

```
CI = μ ± 1.96σ = 50000 ± 1.96(5000)
CI = 50000 ± 9800
CI = [40,200, 59,800] tonnes/year
```

Answer: (a) $P(X > 60,000) = 2.28\%$, (b) 95% CI = [40,200, 59,800] tonnes/year.

Example 2: Lognormal Distribution

Given: Recycling efficiency η is lognormally distributed with: - $\ln(\eta) \sim N(\mu_{ln} = -0.5, \sigma_{ln} = 0.3)$

Find: (a) Mean efficiency $E[\eta]$, (b) Median efficiency.

Solution:

Step 1: Mean of lognormal distribution.

$E[\eta] = \exp(\mu ln + \sigma_{ln}^2/2)$

$E[\eta] = \exp(-0.5 + 0.3^2/2)$

$E[\eta] = \exp(-0.5 + 0.045)$

$E[\eta] = \exp(-0.455) = 0.634$

Step 2: Median of lognormal distribution.

$\text{Median}[\eta] = \exp(\mu_{ln}) = \exp(-0.5) = 0.606$

Step 3: Note: For lognormal, median < mean due to right skew.

Answer: (a) Mean efficiency = 63.4%, (b) Median efficiency = 60.6%.

Example 3: Uncertainty Propagation - Addition

Given: Two independent plastic flows: - $F_1 \sim N(\mu_1 = 100, \sigma_1 = 10)$ tonnes/year - $F_2 \sim N(\mu_2 = 150, \sigma_2 = 15)$ tonnes/year

Find: Mean and standard deviation of total flow $F_{total} = F_1 + F_2$.

Solution:

Step 1: Mean of sum.

$E[F_{total}] = E[F_1] + E[F_2] = 100 + 150 = 250$ tonnes/year

Step 2: Variance of sum (independent variables).

$Var(F_{total}) = Var(F_1) + Var(F_2) = \sigma_1^2 + \sigma_2^2$

$Var(F_{total}) = 10^2 + 15^2 = 100 + 225 = 325$

Step 3: Standard deviation.

$\sigma_{total} = \sqrt{325} = 18.0$ tonnes/year

Answer: $F_{total} \sim N(250, 18.0)$ **tonnes/year.**

Example 4: Uncertainty Propagation - Multiplication

Given: Plastic consumption $C = 1000 \pm 50$ tonnes/year (normal distribution). Recycling rate $RR = 0.30 \pm 0.05$ (normal distribution). Independent variables.

Find: Uncertainty in recycled plastic $R = C \times RR$.

Solution:

Step 1: Calculate mean.

```
E[R] = E[C] × E[RR] = 1000 × 0.30 = 300 tonnes/year
```

Step 2: Apply uncertainty propagation formula.

$\sigma_R^2 = (\partial R/\partial C)^2 \sigma_C^2 + (\partial R/\partial RR)^2 \sigma_{RR}^2$

Step 3: Calculate partial derivatives.

```
∂R/∂C = RR = 0.30
∂R/∂RR = C = 1000
```

Step 4: Calculate variance.

$\sigma_R^2 = (0.30)^2 (50)^2 + (1000)^2 (0.05)^2$

$\sigma_R^2 = 0.09 \times 2500 + 1000000 \times 0.0025$

$\sigma_R^2 = 225 + 2500 = 2725$

Step 5: Calculate standard deviation.

$\sigma_R = \sqrt{2725} = 52.2$ tonnes/year

Answer: R = 300 ± 52.2 tonnes/year (17.4% relative uncertainty).

Example 5: Triangular Distribution

Given: Expert estimates for plastic leakage: - Minimum: a = 1000 tonnes/year - Most likely: c = 2000 tonnes/year - Maximum: b = 4000 tonnes/year

Find: Mean and standard deviation of triangular distribution.

Solution:

Step 1: Mean of triangular distribution.

```
E[X] = (a + b + c) / 3
E[X] = (1000 + 4000 + 2000) / 3 = 7000 / 3 = 2333 tonnes/year
```

Step 2: Variance of triangular distribution.

```
Var(X) = (a² + b² + c² − ab − ac − bc) / 18
```

Step 3: Calculate terms.

```
a² = 1000000
b² = 16000000
c² = 4000000
ab = 4000000
ac = 2000000
bc = 8000000
```

Step 4: Calculate variance.

```
Var(X) = (1000000 + 16000000 + 4000000 - 4000000 - 2000000 - 8000000) / 18
Var(X) = 7000000 / 18 = 388889
```

Step 5: Standard deviation.

```
σ = √388889 = 624 tonnes/year
```

Answer: Mean = 2333 tonnes/year, Standard deviation = 624 tonnes/year.

PRACTICE PROBLEMS

Problem 1: Plastic production P ~ N(500, 50) tonnes/year and waste W ~ N(450, 60) tonnes/year are independent. Calculate the mean and standard deviation of stock change $\Delta S = P - W$.

Problem 2: A lognormal distribution has median m = 100 and 95th percentile p_{95} = 300. Calculate the parameters μ_{ln} and σ_{ln}.

Problem 3: For the function $Y = X_1 / X_2$ where $X_1 = 1000 \pm 100$ and $X_2 = 50 \pm 5$ (independent normals), use uncertainty propagation to estimate σ_Y.

Problem 4: Prove that for independent random variables X and Y, Var(X + Y) = Var(X) + Var(Y).

Problem 5: A uniform distribution on [a, b] has mean $\mu = 50$ and variance $\sigma^2 = 75$. Calculate a and b.

Chapter 7: STATISTICAL METHODS AND MONTE CARLO SIMULATION

7.1 Introduction

Statistical methods enable analysis of plastic accounting data, hypothesis testing, and validation of mass balance models. Monte Carlo simulation provides a powerful approach for propagating uncertainties through complex nonlinear models.

7.2 Descriptive Statistics

For sample $\{x_1, x_2, \ldots, x_n\}$:

Sample mean:

$\bar{x} = (1/n) \Sigma_i x_i$

Sample variance:

$s^2 = (1/(n-1)) \Sigma_i (x_i - \bar{x})^2$

Sample standard deviation:

$s = \sqrt{s^2}$

7.3 Hypothesis Testing

Theorem 7.1 (t-Test for Mean)

To test $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$:

Test statistic:

$t = (\bar{x} - \mu_0) / (s / \sqrt{n})$

Under H_0, t follows Student's t-distribution with n-1 degrees of freedom.

Reject H_0 if $|t| > t_{\alpha/2, n-1}$ where α is significance level.

7.4 Monte Carlo Simulation

Algorithm 7.1 (Monte Carlo Uncertainty Propagation)

To propagate uncertainty through $Y = f(X_1, X_2, \ldots, X_n)$:

1. For i = 1 to N (e.g., N = 10,000):
   - Sample $x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}$ from their distributions
   - Calculate $y^{(i)} = f(x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)})$
2. Analyze $\{y^{(1)}, y^{(2)}, \ldots, y^{(N)}\}$ to obtain distribution of Y

WORKED EXAMPLES

Example 1: Sample Statistics

Given: Plastic waste measurements (tonnes/day): {45, 52, 48, 50, 55, 47, 51}

Find: Sample mean, variance, and standard deviation.

Solution:

Step 1: Calculate sample mean.

```
x̄ = (45 + 52 + 48 + 50 + 55 + 47 + 51) / 7
x̄ = 348 / 7 = 49.71 tonnes/day
```

Step 2: Calculate deviations and squared deviations.

```
(45 - 49.71)² = 22.18
(52 - 49.71)² = 5.24
(48 - 49.71)² = 2.92
(50 - 49.71)² = 0.08
(55 - 49.71)² = 27.98
(47 - 49.71)² = 7.34
(51 - 49.71)² = 1.66
Sum = 67.40
```

Step 3: Calculate sample variance.

```
s² = 67.40 / (7-1) = 67.40 / 6 = 11.23 (tonnes/day)²
```

Step 4: Calculate sample standard deviation.

```
s = √11.23 = 3.35 tonnes/day
```

Answer: Mean = 49.71 tonnes/day, Variance = 11.23, Std Dev = 3.35 tonnes/day.

Example 2: Confidence Interval

Given: Sample from Example 1: $n = 7$, $\bar{x} = 49.71$, $s = 3.35$

Find: 95% confidence interval for population mean.

Solution:

Step 1: Find t-value for 95% CI with df = 6.

$t_{0.025,\,6} = 2.447$ (from t-table)

Step 2: Calculate standard error.

```
SE = s / √n = 3.35 / √7 = 3.35 / 2.646 = 1.27
```

Step 3: Calculate margin of error.

```
ME = t × SE = 2.447 × 1.27 = 3.11
```

Step 4: Construct confidence interval.

```
CI = x̄ ± ME = 49.71 ± 3.11
CI = [46.60, 52.82] tonnes/day
```

Answer: 95% CI = [46.60, 52.82] tonnes/day.

Example 3: Hypothesis Test

Given: A city claims average plastic waste is 50 tonnes/day. Sample data: $n = 7$, $\bar{x} = 49.71$, $s = 3.35$.

Find: Test $H_0: \mu = 50$ vs $H_1: \mu \neq 50$ at $\alpha = 0.05$.

Solution:

Step 1: Calculate test statistic.

```
t = (x̄ − μ₀) / (s / √n)
t = (49.71 − 50) / (3.35 / √7)
t = −0.29 / 1.27 = −0.228
```

Step 2: Find critical value.

$t_{0.025, 6} = \pm 2.447$

Step 3: Compare.

```
|t| = 0.228 < 2.447
```

Step 4: Decision.

```
Fail to reject H₀
```

Answer: Insufficient evidence to reject the claim that mean = 50 tonnes/day ($p > 0.05$).

Example 4: Monte Carlo for Mass Balance

Given: - Production: $P \sim N(1000, 50)$ - Imports: $I \sim N(200, 30)$ - Exports: $E \sim N(150, 20)$ - Waste: $W \sim N(950, 60)$

All independent. Calculate stock change $\Delta S = P + I - E - W$.

Find: Mean and 95% CI for ΔS using Monte Carlo ($N = 10{,}000$).

Solution:

Step 1: Analytical mean (for verification).

```
E[ΔS] = E[P] + E[I] − E[E] − E[W]
E[ΔS] = 1000 + 200 − 150 − 950 = 100
```

Step 2: Analytical variance.

```
Var(ΔS) = Var(P) + Var(I) + Var(E) + Var(W)
Var(ΔS) = 50² + 30² + 20² + 60²
```

```
Var(ΔS) = 2500 + 900 + 400 + 3600 = 7400
σ_ΔS = √7400 = 86.0
```

Step 3: Monte Carlo simulation (pseudocode).

For i = 1 to 10000:

$P_i = random_{normal}(1000, 50)$

$I_i = random_{normal}(200, 30)$

$E_i = random_{normal}(150, 20)$

$W_i = random_{normal}(950, 60)$

$\Delta S_i = P_i + I_i - E_i - W_i$

Step 4: Analyze results (typical output).

```
Mean(ΔS) ≈ 100.2
Std(ΔS) ≈ 85.8
95% CI ≈ [-68, 268]
```

Answer: ΔS ≈ 100 ± 86, 95% CI ≈ [-68, 268] (Monte Carlo confirms analytical result).

Example 5: Regression Analysis

Given: Data on plastic consumption C (Mt/year) vs GDP G (trillion $):

```
G: [1.0, 1.5, 2.0, 2.5, 3.0]
C: [5, 8, 10, 13, 15]
```

Find: Linear regression C = a + bG and R^2.

Solution:

Step 1: Calculate means.

```
Ḡ = (1.0 + 1.5 + 2.0 + 2.5 + 3.0) / 5 = 2.0
C̄ = (5 + 8 + 10 + 13 + 15) / 5 = 10.2
```

Step 2: Calculate slope b.

$b = \Sigma(G_i - \bar{G})(C_i - \bar{C}) / \Sigma(G_i - \bar{G})^2$

Numerator:

$(-1.0)(-5.2) + (-0.5)(-2.2) + (0)(-0.2) + (0.5)(2.8) + (1.0)(4.8)$

$= 5.2 + 1.1 + 0 + 1.4 + 4.8 = 12.5$

Denominator:

$(-1.0)^2 + (-0.5)^2 + 0^2 + (0.5)^2 + (1.0)^2$

$= 1.0 + 0.25 + 0 + 0.25 + 1.0 = 2.5$

$b = 12.5 / 2.5 = 5.0$

Step 3: Calculate intercept a.

$a = \bar{C} - b\,\bar{G} = 10.2 - 5.0(2.0) = 0.2$

Step 4: Regression equation.

$C = 0.2 + 5.0\,G$

Step 5: Calculate R^2.

$SST = \Sigma(Ci - \bar{C})^2 = 5.2^2 + 2.2^2 + 0.2^2 + 2.8^2 + 4.8^2 = 62.8$

$SSR = b^2\,\Sigma(Gi - \bar{G})^2 = 5.0^2 \times 2.5 = 62.5$

$R^2 = SSR / SST = 62.5 / 62.8 = 0.995$

Answer: $C = 0.2 + 5.0G$, $R^2 = 0.995$ (excellent fit).

PRACTICE PROBLEMS

Problem 1: A sample of 15 recycling facilities has mean efficiency $\bar{x} = 0.75$ and standard deviation $s = 0.08$. Construct a 99% confidence interval for the population mean efficiency.

Problem 2: Test whether recycling rate has increased from historical value of 0.25. New sample: $n = 20$, $\bar{x} = 0.28$, $s = 0.05$. Use $\alpha = 0.05$.

Problem 3: Design a Monte Carlo simulation to estimate the distribution of Material Circularity Indicator $MCI = 1 - (V + W)/2C$, where $V \sim N(700, 50)$, $W \sim N(600, 60)$, $C \sim N(1000, 80)$. Run $N = 10,000$ iterations.

Problem 4: For the regression in Example 5, calculate the predicted consumption at $G = 3.5$ trillion \$ and the residual sum of squares (RSS).

Problem 5: Prove that the sample variance $s^2 = \Sigma(x_i - \bar{x})^2 / (n-1)$ is an unbiased estimator of population variance σ^2, i.e., $E[s^2] = \sigma^2$.

Chapter 8: NETWORK FLOW ANALYSIS

8.1 Introduction

Plastic systems can be represented as networks with nodes (processes) and edges (flows). Network flow analysis provides algorithms for optimizing flows, finding bottlenecks, and analyzing system structure.

8.2 Graph Theory Foundations

Definition 8.1 (Directed Graph)

A directed graph $G = (V, E)$ consists of: - $V = \{v_1, v_2, \ldots, v_n\}$: set of vertices (processes) - $E \subseteq V \times V$: set of directed edges (flows)

Definition 8.2 (Flow Network)

A flow network is a directed graph with: - Capacity function $c: E \to \mathbb{R}^+$ (maximum flow on each edge) - Source node $s \in V$ (net outflow) - Sink node $t \in V$ (net inflow)

8.3 Conservation of Flow

Theorem 8.1 (Flow Conservation)

For all nodes $v \in V \ \{s, t\}$:

$$\Sigma_{u:(u,v)\in E} f(u,v) = \Sigma_{w:(v,w)\in E} f(v,w)$$

Inflow equals outflow at all intermediate nodes.

8.4 Maximum Flow Problem

Theorem 8.2 (Max-Flow Min-Cut Theorem)

The maximum flow from s to t equals the minimum capacity of any cut separating s from t.

WORKED EXAMPLES

Example 1: Simple Flow Network

Given: Network with 4 nodes: Source (S), Process A, Process B, Sink (T). Edges and capacities: - S → A: 100 - S → B: 80 - A → T: 70 - B → T: 90 - A → B: 50

Find: Maximum flow from S to T.

Solution:

Step 1: Identify possible paths from S to T.

```
Path 1: S → A → T (capacity limited by min(100, 70) = 70)
Path 2: S → B → T (capacity limited by min(80, 90) = 80)
Path 3: S → A → B → T (capacity limited by min(100, 50, 90) = 50)
```

Step 2: Allocate flow.

```
Send 70 on Path 1: S → A → T
Send 80 on Path 2: S → B → T
Remaining capacity on A → B: 50
```

Step 3: Check if additional flow possible via Path 3.

```
S → A capacity used: 70 (30 remaining)
A → B capacity: 50
B → T capacity used: 80 (10 remaining)
Can send min(30, 50, 10) = 10 more
```

Step 4: Total maximum flow.

```
Max flow = 70 + 80 + 10 = 160
```

Answer: Maximum flow = 160 units.

Example 2: Flow Conservation Check

Given: Node A has: - Inflows: 50 (from S), 30 (from B) - Outflows: 60 (to C), 20 (to D)

Find: Verify flow conservation.

Solution:

Step 1: Calculate total inflow.

$\text{Inflow}_{\text{total}} = 50 + 30 = 80$

Step 2: Calculate total outflow.

$\text{Outflow}_{\text{total}} = 60 + 20 = 80$

Step 3: Check conservation.

$\text{Inflow}_{\text{total}} = \text{Outflow}_{\text{total}} = 80$ ✓

Answer: Flow conservation is satisfied.

Example 3: Minimum Cut

Given: Network from Example 1. Find minimum cut separating S from T.

Solution:

Step 1: List all possible cuts (partitions of nodes into {S-side, T-side}).

Cut 1: {S} | {A, B, T}

```
Capacity = c(S,A) + c(S,B) = 100 + 80 = 180
```

Cut 2: {S, A} | {B, T}

```
Capacity = c(S,B) + c(A,T) + c(A,B) = 80 + 70 + 50 = 200
```

Cut 3: {S, B} | {A, T}

```
Capacity = c(S,A) + c(B,T) = 100 + 90 = 190
```

Cut 4: {S, A, B} | {T}

```
Capacity = c(A,T) + c(B,T) = 70 + 90 = 160
```

Step 2: Identify minimum.

```
Min cut capacity = 160 (Cut 4)
```

Step 3: Verify max-flow min-cut theorem.

```
Max flow = 160 = Min cut capacity ✓
```

Answer: Minimum cut = {S, A, B} | {T} with capacity 160.

Example 4: Bottleneck Identification

Given: Linear process chain: S → A → B → C → T Capacities: c(S,A) = 100, c(A,B) = 80, c(B,C) = 120, c(C,T) = 90

Find: System bottleneck.

Solution:

Step 1: For linear chain, max flow = min capacity.

```
Max flow = min(100, 80, 120, 90) = 80
```

Step 2: Bottleneck is edge with minimum capacity.

```
Bottleneck: A → B with capacity 80
```

Step 3: Interpret: Upgrading A → B capacity would increase system throughput.

Answer: Bottleneck is A → B (capacity 80); limits system to 80 units/time.

Example 5: Multi-Commodity Flow

Given: Two plastic types (PET and HDPE) flow through shared network. Node A has capacity 100 total. - PET flow: 60 - HDPE flow: 50

Find: Is the flow feasible?

Solution:

Step 1: Check capacity constraint.

```
Total flow = 60 + 50 = 110
Node capacity = 100
```

Step 2: Compare.

```
110 > 100: Infeasible
```

Step 3: Calculate required reduction.

```
Excess = 110 - 100 = 10
```

Answer: Flow is infeasible; must reduce total flow by 10 units.

PRACTICE PROBLEMS

Problem 1: For a network with nodes {S, A, B, C, T} and edges S→A (50), S→B (60), A→C (40), B→C (50), C→T (70), find the maximum flow from S to T using Ford-Fulkerson algorithm.

Problem 2: Prove that in any flow network, the total flow out of the source equals the total flow into the sink.

Problem 3: A recycling network has three collection points feeding two processing facilities. Formulate this as a minimum-cost flow problem with collection costs and processing capacities.

Problem 4: Calculate the number of possible cuts in a network with n nodes. For n = 5, enumerate all cuts and find the minimum cut for a given capacity matrix.

Problem 5: Design a network flow model for a city's plastic waste management system with 5 collection zones, 3 sorting facilities, 2 recycling plants, and 1 landfill. Include capacity constraints and flow conservation equations.

Chapter 9: STOCHASTIC MODELING FOR PLASTIC FLOWS

9.1 Introduction

Plastic flows exhibit randomness due to variability in consumption patterns, product lifetimes, and waste generation. Stochastic models capture this inherent variability and enable probabilistic forecasting.

9.2 Stochastic Differential Equations

Definition 9.1 (Wiener Process)

A Wiener process W(t) (Brownian motion) satisfies: - $W(0) = 0$ - $W(t) - W(s) \sim N(0, t-s)$ for $t > s$ - Independent increments

Definition 9.2 (Stochastic Differential Equation)

A stochastic plastic stock model:

```
dS = μ(S, t) dt + σ(S, t) dW
```

where μ is drift, σ is diffusion coefficient, dW is Wiener increment.

9.3 Markov Chains

Definition 9.3 (Discrete-Time Markov Chain)

A sequence $\{X_n\}$ where:

$P(X_{n+1} = j \mid X_n = i, X_{n-1}, ..., X_0) = P(X_{n+1} = j \mid X_n = i) = p_{ij}$

Transition matrix P with elements p_{ij}.

9.4 Poisson Processes

Definition 9.4 (Poisson Process)

Count process N(t) with rate λ:

```
P(N(t) = k) = (λt)^k exp(-λt) / k!
```

Models random arrivals (e.g., waste generation events).

WORKED EXAMPLES

Example 1: Geometric Brownian Motion

Given: Plastic production follows:

```
dP/P = μ dt + σ dW
```

with $\mu = 0.03$/year, $\sigma = 0.10$/year, $P(0) = 400$ Mt/year.

Find: Expected production P(10) after 10 years.

Solution:

Step 1: Solution to geometric Brownian motion:

$P(t) = P(0) \exp((\mu - \sigma^2/2)t + \sigma W(t))$

Step 2: Expected value:

```
E[P(t)] = P(0) exp(μ t)
```

Step 3: Calculate E[P(10)].

```
E[P(10)] = 400 × exp(0.03 × 10)
E[P(10)] = 400 × exp(0.3)
E[P(10)] = 400 × 1.350 = 540 Mt/year
```

Step 4: Variance:

$\mathrm{Var}(P(t)) = P(0)^2 \exp(2\mu t) [\exp(\sigma^2 t) - 1]$

$\mathrm{Var}(P(10)) = 400^2 \times \exp(0.6) \times [\exp(0.1) - 1]$

$\mathrm{Var}(P(10)) = 160000 \times 1.822 \times 0.105 = 30{,}610$

$\sigma_P(10) = \sqrt{30610} = 175$ Mt/year

Answer: $E[P(10)] = 540$ Mt/year, $\sigma_P(10) = 175$ Mt/year.

Example 2: Two-State Markov Chain

Given: Recycling system has two states: High efficiency (H) and Low efficiency (L). Transition matrix:

```
**P** = [ 0.9   0.1 ]   (H → H: 0.9, H → L: 0.1)
        [ 0.3   0.7 ]   (L → H: 0.3, L → L: 0.7)
```

Find: Steady-state probabilities.

Solution:

Step 1: Steady state satisfies $\pi = \pi$ **P**.

$\pi_H = 0.9\ \pi_H + 0.3\ \pi_L$

$\pi_L = 0.1\ \pi_H + 0.7\ \pi_L$

Step 2: Also $\pi_H + \pi_L = 1$.

Step 3: From first equation:

$\pi_H - 0.9\ \pi_H = 0.3\ \pi_L$

$0.1\ \pi_H = 0.3\ \pi_L$

$\pi_H = 3\ \pi_L$

Step 4: Substitute into normalization:

$3\ \pi_L + \pi_L = 1$

$4\ \pi_L = 1$

$\pi_L = 0.25$

$\pi_H = 0.75$

Answer: Steady state: 75% in High efficiency, 25% in Low efficiency.

Example 3: Poisson Waste Generation

Given: Waste collection events follow Poisson process with rate $\lambda = 2$ events/day.

Find: (a) Probability of exactly 3 events in one day, (b) Expected time until next event.

Solution:

Step 1: Poisson probability for $k = 3$, $t = 1$ day.

```
P(N(1) = 3) = (λt)³ exp(-λt) / 3!
P(N(1) = 3) = (2×1)³ exp(-2×1) / 6
P(N(1) = 3) = 8 × 0.135 / 6 = 0.180
```

Step 2: Inter-arrival time is exponentially distributed:

```
E[T] = 1/λ = 1/2 = 0.5 days = 12 hours
```

Answer: (a) P(3 events) = 18.0%, (b) Expected time to next event = 12 hours.

Example 4: Random Walk Stock Model

Given: Stock changes by ±10 tonnes each year with equal probability. Initial stock $S(0) = 100$ tonnes.

Find: Expected stock and variance after 5 years.

Solution:

Step 1: This is a simple random walk.

$S(n) = S(0) + \Sigma_{i=1}$^n X_i

where $X_i \in \{-10, +10\}$ with $P(X_i = +10) = P(X_i = -10) = 0.5$.

Step 2: Expected value of each step:

$E[X_i] = 0.5(+10) + 0.5(-10) = 0$

Step 3: Expected stock:

$E[S(5)] = S(0) + \Sigma\ E[X_i] = 100 + 5(0) = 100$ tonnes

Step 4: Variance of each step:

$Var(Xi) = E[X_i^2] - (E[Xi])^2 = 0.5(100) + 0.5(100) - 0 = 100$

Step 5: Variance of stock (independent steps):

$Var(S(5)) = \Sigma\ Var(X_i) = 5 \times 100 = 500$

$\sigma_S(5) = \sqrt{500} = 22.4$ tonnes

Answer: $E[S(5)] = 100$ tonnes, $\sigma_S(5) = 22.4$ tonnes.

Example 5: Ornstein-Uhlenbeck Process

Given: Recycling rate r(t) mean-reverts to long-term average $\bar{r} = 0.30$:

```
dr = θ(r̄ − r) dt + σ dW
```

with $\theta = 0.5$/year, $\sigma = 0.05$/year, $r(0) = 0.20$.

Find: Expected recycling rate after 2 years.

Solution:

Step 1: Solution to OU process:

```
r(t) = r̄ + (r(0) − r̄) exp(−θt)
```

Step 2: Calculate r(2).

```
r(2) = 0.30 + (0.20 − 0.30) exp(−0.5 × 2)
r(2) = 0.30 + (−0.10) exp(−1.0)
r(2) = 0.30 − 0.10 × 0.368
r(2) = 0.30 − 0.0368 = 0.263
```

Answer: Expected recycling rate after 2 years = 26.3%.

PRACTICE PROBLEMS

Problem 1: For geometric Brownian motion $dP/P = 0.04\ dt + 0.12\ dW$ with $P(0) = 500$, calculate the 95% confidence interval for $P(5)$.

Problem 2: A three-state Markov chain has transition matrix **P** = [[0.7, 0.2, 0.1], [0.3, 0.5, 0.2], [0.1, 0.3, 0.6]]. Find the steady-state distribution.

Problem 3: Plastic leakage events follow a Poisson process with rate $\lambda = 0.5$ events/month. Calculate the probability of zero leakage events in a 6-month period.

Problem 4: Derive the variance formula $Var(S(n)) = n\ Var(X)$ for a random walk $S(n) = S(0) + \Sigma X_i$ with independent identically distributed X_i.

Problem 5: For an Ornstein-Uhlenbeck process with parameters $\theta = 1.0$, $\bar{r} = 0.25$, $\sigma = 0.08$, calculate the long-term variance: $Var_\infty(r) = \sigma^2/(2\theta)$.

PART III: ORGANIZATIONAL PLASTIC ACCOUNTING

Chapter 10: SCOPE 1 PLASTIC USE - DIRECT SOURCES

10.1 Introduction

Scope 1 plastic use encompasses all plastic directly consumed, used, or generated as waste within an organization's owned or controlled operations. This framework is adapted from the GHG Protocol's Scope 1 emissions but applied to material flows rather than atmospheric emissions.

10.2 Scope 1 Boundaries

Definition 10.1 (Scope 1 Plastic)

Scope 1 plastic includes all plastic flows within the organizational boundary that are under direct operational control:

1. **Direct plastic consumption:** Raw plastic materials (resins, pellets) used in manufacturing
2. **Operational packaging:** Plastic packaging materials used in internal operations
3. **On-site waste generation:** Plastic waste generated within facilities
4. **Direct leakage:** Plastic leakage from facilities to the environment

Theorem 10.1 (Scope 1 Mass Balance)

For organizational boundary Ω with operational control:

$Scope1 = P_{direct} + M_{packaging} + W_{operational}$

where: - P_{direct} = direct plastic consumption (kg/year) - $M_{packaging}$ = operational packaging materials (kg/year) - $W_{operational}$ = on-site waste generation (kg/year)

Proof:

Step 1: Define organizational boundary Ω enclosing all facilities under operational control.

Step 2: Apply mass balance within Ω:

$dS/dt = F_{in} - F_{out}$

Step 3: Scope 1 represents all plastic inputs under direct control:

$$F_{in,Scope1} = P_{direct} + M_{packaging}$$

Step 4: Waste generated on-site is also Scope 1:

$$W_{operational} \subset Scope1$$

Step 5: Total Scope 1 accounting:

$$Scope1 = P_{direct} + M_{packaging} + W_{operational}$$

■

10.3 Quantification Methods

Direct Measurement

- Purchase records for plastic resins
- Inventory tracking systems
- Waste audit data

Estimation Methods

- Production-based estimates
- Material balance calculations
- Statistical sampling

WORKED EXAMPLES

Example 1: Manufacturing Facility Scope 1 Calculation

Given: A manufacturing facility has the following annual data: - Polyethylene (PE) resin purchased: 500 tonnes/year - Polypropylene (PP) resin purchased: 300 tonnes/year - Shrink wrap for internal use: 15 tonnes/year - Pallet wrap for shipping: 25 tonnes/year - Manufacturing scrap: 80 tonnes/year - Defective products scrapped: 20 tonnes/year

Find: Total Scope 1 plastic use.

Solution:

Step 1: Identify direct plastic consumption (raw materials).

$P_{direct} = PE + PP = 500 + 300 = 800$ tonnes/year

Step 2: Identify operational packaging.

$M_{packaging,internal}$ = Shrink wrap = 15 tonnes/year

Note: Pallet wrap for shipping is Scope 3 (downstream transportation), not Scope 1.

Step 3: Identify on-site waste generation.

$W_{operational}$ = Manufacturing scrap + Defective products
$W_{operational} = 80 + 20 = 100$ tonnes/year

Step 4: Calculate total Scope 1.

$Scope1 = P_{direct} + M_{packaging,internal} + W_{operational}$
$Scope1 = 800 + 15 + 100 = 915$ tonnes/year

Step 5: Verify: Waste should not exceed inputs.

$W_{operational}$ (100) < $P_{direct} + M_{packaging}$ (815) ✓

Answer: Scope 1 plastic use = 915 tonnes/year.

Example 2: Multi-Facility Aggregation

Given: A company has three facilities: - Facility A: Scope 1 = 1,200 tonnes/year - Facility B: Scope 1 = 850 tonnes/year - Facility C: Scope 1 = 600 tonnes/year

Inter-facility transfers: Facility A ships 50 tonnes/year of plastic components to Facility B.

Find: Total organizational Scope 1.

Solution:

Step 1: Sum Scope 1 from all facilities.

$$Scope1_{total} = Scope1_A + Scope1_B + Scope1_C$$
$$Scope1_{total} = 1200 + 850 + 600 = 2650 \text{ tonnes/year}$$

Step 2: Check for double counting of inter-facility transfers.

Inter-facility transfers are internal to the organizational boundary and should not be double-counted. They are already included in each facility's Scope 1.

Step 3: Verify no adjustment needed.

The 50 tonnes shipped from A to B: - Counted in A's production (part of 1200) - Received by B (part of 850) - But B's Scope 1 counts B's direct inputs, not transfers from A

If the 50 tonnes were counted in both, we would subtract once:

$$Scope1_{corrected} = 2650 - 50 = 2600 \text{ tonnes/year}$$

However, proper accounting should have B count only its direct purchases, so:

Answer: Scope 1 = 2,650 tonnes/year (assuming no double counting in facility-level data).

Example 3: Temporal Allocation

Given: A facility purchases plastic resins quarterly: - Q1: 200 tonnes - Q2: 250 tonnes - Q3: 180 tonnes - Q4: 270 tonnes

Waste generation is measured monthly and totals 85 tonnes for the year.

Find: (a) Annual Scope 1, (b) Q2 Scope 1 allocation.

Solution:

Step 1: Calculate annual direct consumption.

$$P_{direct,annual} = 200 + 250 + 180 + 270 = 900 \text{ tonnes/year}$$

Step 2: Calculate annual Scope 1.

$Scope1_{annual} = P_{direct} + W_{operational}$

$Scope1_{annual} = 900 + 85 = 985$ tonnes/year

Step 3: Allocate Q2 Scope 1.

Assume waste is proportional to consumption:

$W_{Q2} = (P_{Q2} / P_{annual}) \times W_{annual}$

$W_{Q2} = (250 / 900) \times 85 = 23.6$ tonnes

Step 4: Calculate Q2 Scope 1.

$Scope1_{Q2} = P_{Q2} + W_{Q2} = 250 + 23.6 = 273.6$ tonnes

Answer: (a) Annual Scope 1 = 985 tonnes/year, (b) Q2 Scope 1 = 273.6 tonnes.

Example 4: Leakage Estimation

Given: A facility handles 1,000 tonnes/year of plastic pellets. Industry studies estimate 0.5% pellet loss during handling and transport.

Find: Scope 1 leakage.

Solution:

Step 1: Apply leakage fraction.

$L_{direct} = f_{leak} \times M_{handled}$

where $f_{leak} = 0.005$ (0.5%).

Step 2: Calculate leakage.

$L_{direct} = 0.005 \times 1000 = 5$ tonnes/year

Step 3: Convert to daily rate.

L_{daily} = 5 / 365 = 0.0137 tonnes/day = 13.7 kg/day

Step 4: This leakage is part of Scope 1 (direct operational control).

Answer: Scope 1 direct leakage = 5 tonnes/year or 13.7 kg/day.

Example 5: Scope 1 Intensity Metric

Given: A company has: - Scope 1 plastic use: 2,500 tonnes/year - Annual revenue: $50 million - Production output: 10,000 units/year

Find: (a) Revenue intensity, (b) Production intensity.

Solution:

Step 1: Calculate revenue intensity.

$I_{revenue}$ = Scope1 / Revenue
$I_{revenue}$ = 2500 tonnes / 50 million $ = 0.05 tonnes/$1000
$I_{revenue}$ = 50 kg/$1000 revenue

Step 2: Calculate production intensity.

$I_{production}$ = Scope1 / Output
$I_{production}$ = 2500 tonnes / 10000 units = 0.25 tonnes/unit = 250 kg/unit

Step 3: Interpret: These intensity metrics allow benchmarking and target setting.

Answer: (a) Revenue intensity = 50 kg/$1000, (b) Production intensity = 250 kg/unit.

PRACTICE PROBLEMS

Problem 1: A pharmaceutical manufacturing facility uses 150 tonnes/year of plastic for packaging (bottles, caps) and 80 tonnes/year for laboratory consumables (pipette tips, petri dishes).

Manufacturing waste is 25 tonnes/year. Calculate Scope 1 plastic use and identify which flows are under direct operational control.

Problem 2: A company operates 5 facilities globally. Facilities 1-3 are wholly owned (operational control), Facilities 4-5 are joint ventures with 40% ownership share. Should Facilities 4-5 be included in Scope 1? If so, how should they be accounted for (100% or 40%)?

Problem 3: A facility purchases 1,000 tonnes/year of virgin plastic resin and 200 tonnes/year of recycled plastic resin. Should both be counted in Scope 1? How should recycled content be reported separately?

Problem 4: Monthly plastic consumption varies seasonally: Jan-Mar (80 t/month), Apr-Jun (120 t/month), Jul-Sep (150 t/month), Oct-Dec (100 t/month). Calculate: (a) annual Scope 1, (b) peak month intensity, (c) coefficient of variation.

Problem 5: A facility implements a waste reduction program that reduces manufacturing scrap from 100 tonnes/year to 60 tonnes/year. Raw material consumption remains 800 tonnes/year. Calculate the Scope 1 reduction and explain why waste reduction affects Scope 1 even though inputs are unchanged.

Chapter 11: SCOPE 2 PLASTIC USE - EMBEDDED IN PURCHASED GOODS

11.1 Introduction

Scope 2 plastic represents plastic embedded in goods and services purchased by the organization but not under direct operational control. This includes plastic in purchased components, raw materials (non-plastic products containing plastic), capital equipment, and purchased packaging.

11.2 Scope 2 Quantification Methods

Definition 11.1 (Scope 2 Plastic)

$Scope2 = \Sigma_i (M_{purchased,i} \times f_{plastic,i})$

where: - $M_{purchased,i}$ = mass of purchased item i (kg) - $f_{plastic,i}$ = plastic content fraction of item i (dimensionless)

Theorem 11.1 (Input-Output Method for Scope 2)

Using economic input-output analysis:

$Scope2 = \Sigma_j (E_j \times I_{plastic,j})$

$¿ - I_p lastic, j = plasticintensityofsectorj ¿$where: - E_j = expenditure in economic sector j ()

WORKED EXAMPLES

Example 1: Component Plastic Content

Given: An electronics manufacturer purchases: - 10,000 plastic housings at 0.5 kg each - 50,000 circuit boards (10% plastic by mass, 0.2 kg each) - 5,000 power supplies (5% plastic by mass, 1.5 kg each)

Find: Scope 2 plastic from components.

Solution:

Step 1: Calculate plastic in housings.

$M_{housings} = 10000 \times 0.5 = 5000$ kg

Step 2: Calculate plastic in circuit boards.

$M_{boards,total} = 50000 \times 0.2 = 10000$ kg
$M_{boards,plastic} = 10000 \times 0.10 = 1000$ kg

Step 3: Calculate plastic in power supplies.

$M_{supplies,total} = 5000 \times 1.5 = 7500$ kg
$M_{supplies,plastic} = 7500 \times 0.05 = 375$ kg

Step 4: Sum Scope 2.

```
Scope2 = 5000 + 1000 + 375 = 6375 kg/year = 6.375 tonnes/year
```

Answer: Scope 2 = 6.375 tonnes/year.

Example 2: Input-Output Method

Given: A company's annual expenditures: - Packaging materials: 500,000¿) - Office supplies: 100,000¿) - IT equipment: 200,000¿)

Find: Scope 2 using input-output method.

Solution:

Step 1: Calculate plastic from packaging.

$P_{packaging} = E \times I = 500000 \times 0.8 = 400,000$ kg

Step 2: Calculate plastic from office supplies.

$P_{office} = 100000 \times 0.15 = 15{,}000$ kg

Step 3: Calculate plastic from IT equipment.

$P_{IT} = 200000 \times 0.25 = 50{,}000$ kg

Step 4: Sum Scope 2.

```
Scope2 = 400000 + 15000 + 50000 = 465,000 kg = 465 tonnes/year
```

Answer: Scope 2 = 465 tonnes/year.

Example 3: Hybrid Method

Given: - Direct supplier data: 80% of purchases, plastic content known = 350 tonnes - Remaining 20%: Expenditure = $150{,}000, average\, I_p lastic = 0.4\, kg/¿$

Find: Total Scope 2 using hybrid method.

Solution:

Step 1: Scope 2 from direct data (80%).

$Scope2_{direct} = 350$ tonnes

Step 2: Scope 2 from input-output (20%).

$Scope2_{IO} = E \times I = 150000 \times 0.4 = 60{,}000$ kg $= 60$ tonnes

Step 3: Total Scope 2.

$Scope2_{total} = 350 + 60 = 410$ tonnes/year

Answer: Scope 2 = 410 tonnes/year (hybrid method).

Example 4: Purchased Packaging Attribution

Given: A retailer purchases products with packaging: - Product A: 10,000 units, 50g packaging each - Product B: 5,000 units, 120g packaging each - Product C: 20,000 units, 30g packaging each

Find: Scope 2 from purchased packaging.

Solution:

Step 1: Calculate packaging for Product A.

$M_A = 10000 \times 0.050 = 500$ kg

Step 2: Calculate packaging for Product B.

$M_B = 5000 \times 0.120 = 600$ kg

Step 3: Calculate packaging for Product C.

$M_C = 20000 \times 0.030 = 600$ kg

Step 4: Total Scope 2 packaging.

$$\text{Scope2}_{\text{packaging}} = 500 + 600 + 600 = 1700 \text{ kg} = 1.7 \text{ tonnes/year}$$

Answer: Scope 2 from purchased packaging = 1.7 tonnes/year.

Example 5: Capital Goods Allocation

Given: A company purchases machinery with 500 kg plastic content for $1,000,000. Expected lifetime: 10 years.

Find: Annual Scope 2 allocation for capital goods.

Solution:

Step 1: Total plastic in capital goods.

$M_{capital} = 500$ kg

Step 2: Allocate over lifetime (straight-line).

$Scope2_{annual} = M_{capital}$ / Lifetime = 500 / 10 = 50 kg/year

Step 3: Alternative: Count full amount in year of purchase.

$Scope2_{purchase\_year}$ = 500 kg

Step 4: Recommendation: Use amortization for consistency with financial accounting.

Answer: Annual Scope 2 (amortized) = 50 kg/year, or 500 kg in purchase year.

PRACTICE PROBLEMS

Problem 1: A food manufacturer purchases 1,000 tonnes of ingredients annually. Packaging for these ingredients contains 15 tonnes of plastic. Is this Scope 1, Scope 2, or Scope 3? Justify your answer.

Problem 2: Calculate Scope 2 for a company purchasing: (a) 500 laptops at 1.2 kg plastic each, (b) 10,000 pens at 5g plastic each, (c) 200 office chairs at 3 kg plastic each.

Problem 3: A company spends 2 *millionannuallyonpurchasedgoods*. *Ifaverageplasticintensityis* 0.35 *kg*/¿, estimate Scope 2. What are the limitations of this approach?

Problem 4: Distinguish between Scope 1 and Scope 2 for: (a) plastic resin purchased for manufacturing, (b) plastic components purchased for assembly, (c) plastic packaging purchased for shipping products.

Problem 5: A company wants to reduce Scope 2 plastic by 20%. Propose three strategies and calculate the required reduction if current Scope 2 = 800 tonnes/year.

Chapter 12: SCOPE 3 PLASTIC USE - VALUE CHAIN

12.1 Introduction

Scope 3 encompasses all plastic in the organization's value chain beyond Scope 1 and 2. This includes upstream supply chain impacts and downstream product use and end-of-life.

12.2 Scope 3 Categories

Following GHG Protocol structure, adapted for plastic:

Upstream (Categories 1-8): 1. Purchased goods and services 2. Capital goods 3. Fuel and energy-related activities 4. Upstream transportation and distribution 5. Waste generated in operations 6. Business travel 7. Employee commuting 8. Upstream leased assets

Downstream (Categories 9-15): 9. Downstream transportation and distribution 10. Processing of sold products 11. Use of sold products 12. End-of-life treatment of sold products 13. Downstream leased assets 14. Franchises 15. Investments

Theorem 12.1 (Scope 3 Completeness)

$PF_{total} = Scope1 + Scope2 + Scope3$

with no double counting across scopes.

WORKED EXAMPLES

Example 1: Category 4 - Upstream Transportation Packaging

Given: A company receives 5,000 shipments annually. Average packaging per shipment: 2 kg plastic.

Find: Scope 3 Category 4 plastic.

Solution:

Step 1: Calculate total packaging.

$M_{transport} = N_{shipments} \times M_{per\_shipment}$

$M_{transport} = 5000 \times 2 = 10{,}000$ kg = 10 tonnes/year

Step 2: This is Scope 3 (not owned by company, used by suppliers).

Answer: Scope 3 Category 4 = 10 tonnes/year.

Example 2: Category 12 - End-of-Life of Sold Products

Given: A company sells 100,000 products annually, each containing 1.5 kg plastic. Product lifetime: 5 years.

Find: Annual Scope 3 Category 12 plastic waste.

Solution:

Step 1: Calculate annual sales plastic.

$M_{sales} = 100000 \times 1.5 = 150{,}000$ kg/year = 150 tonnes/year

Step 2: In steady state, waste generation equals sales.

$W_{EoL} = M_{sales} = 150$ tonnes/year

Step 3: This is Scope 3 Category 12 (downstream end-of-life).

Answer: Scope 3 Category 12 = 150 tonnes/year.

Example 3: Category 11 - Use of Sold Products

Given: A company sells plastic containers. Customers purchase replacement lids (sold separately): 50,000 lids/year at 50g each.

Find: Scope 3 Category 11 plastic.

Solution:

Step 1: Calculate plastic in replacement parts.

$M_{lids} = 50000 \times 0.050 = 2{,}500$ kg = 2.5 tonnes/year

Step 2: This is Scope 3 Category 11 (plastic consumed during use phase).

Answer: Scope 3 Category 11 = 2.5 tonnes/year.

Example 4: Category 9 - Downstream Transportation

Given: A manufacturer ships products to retailers. Shipping packaging: 25 tonnes/year plastic.

Find: Scope classification.

Solution:

Step 1: Determine who owns/controls the packaging.

If manufacturer provides packaging: Scope 1 or 3? - If packaging is for manufacturer's products being shipped: Scope 3 Category 9

Step 2: Classify as Scope 3 Category 9 (downstream transportation).

Answer: Scope 3 Category 9 = 25 tonnes/year.

Example 5: Total Value Chain Footprint

Given: - Scope 1: 500 tonnes/year - Scope 2: 300 tonnes/year - Scope 3 upstream (Cat 1-8): 1,200 tonnes/year - Scope 3 downstream (Cat 9-15): 2,000 tonnes/year

Find: (a) Total footprint, (b) Scope 3 percentage.

Solution:

Step 1: Calculate total Scope 3.

$Scope3 = 1200 + 2000 = 3200$ tonnes/year

Step 2: Calculate total footprint.

$PF_{total} = Scope1 + Scope2 + Scope3$

$PF_{total} = 500 + 300 + 3200 = 4000$ tonnes/year

Step 3: Calculate Scope 3 percentage.

$Scope3_{\%} = (3200 / 4000) \times 100\% = 80\%$

Answer: (a) Total footprint = 4,000 tonnes/year, (b) Scope 3 = 80% of total.

PRACTICE PROBLEMS

Problem 1: Classify the following as Scope 1, 2, or 3: (a) Plastic in products sold to customers, (b) Plastic waste from company cafeteria, (c) Plastic in supplier’s packaging, (d) Plastic in employee personal vehicles.

Problem 2: A beverage company sells 1 billion bottles/year (30g plastic each). Calculate Scope 3 Category 12 assuming all bottles become waste after single use.

Problem 3: Why is Scope 3 typically much larger than Scope 1+2 for consumer goods companies? Provide quantitative reasoning.

Problem 4: A company’s Scope 3 is 5,000 tonnes/year. Management wants to reduce it by 30% over 5 years. What annual reduction rate is required? (Hint: Use exponential decay model)

Problem 5: Design a Scope 3 screening methodology to identify the top 3 categories contributing 80% of Scope 3 plastic. What data would you need?

Chapter 13: PLASTIC FOOTPRINT CALCULATION METHODS

13.1 Introduction

Plastic footprinting quantifies the total plastic attributable to a product, service, or organization across its lifecycle. This chapter provides rigorous methods for calculating footprints at different scales.

13.2 Product-Level Footprinting

Definition 13.1 (Product Plastic Footprint)

$PPF = M_{product} + M_{packaging} + M_{manufacturing} + M_{transport} + M_{EoL}$

where each term represents plastic mass (kg) attributable to the product.

Theorem 13.1 (Functional Unit Normalization)

For comparative footprinting:

$PPF_{normalized} = PPF / FU$

where FU is the functional unit (e.g., per liter, per use, per year of service).

WORKED EXAMPLES

Example 1: Beverage Bottle Footprint

Given: A 1-liter PET bottle: - Bottle: 30g PET - Cap: 3g HDPE - Label: 2g PP - Manufacturing scrap rate: 5% - Transport packaging: 1g per bottle

Find: Total product plastic footprint.

Solution:

Step 1: Calculate product plastic.

$M_{product} = 30 + 3 + 2 = 35g$

Step 2: Account for manufacturing scrap.

$M_{manufacturing} = M_{product} \times (scrap_{rate} / (1 - scrap_{rate}))$
$M_{manufacturing} = 35 \times (0.05 / 0.95) = 1.84g$

Step 3: Add transport packaging.

$M_{transport} = 1g$

Step 4: Total footprint.

```
PPF = 35 + 1.84 + 1 = 37.84g ≈ 38g
```

Step 5: Normalize per liter.

$PPF_{normalized} = 38g / 1L = 38 g/L$

Answer: Product plastic footprint = 38g per 1-liter bottle.

Example 2: Allocation by Mass

Given: A production line makes two products: - Product A: 10,000 units/year, 500g plastic each - Product B: 5,000 units/year, 1,200g plastic each

Shared manufacturing waste: 800 kg/year

Find: Allocate waste to each product.

Solution:

Step 1: Calculate total product plastic.

$M_A = 10000 \times 0.5 = 5{,}000$ kg/year
$M_B = 5000 \times 1.2 = 6{,}000$ kg/year
$M_{total} = 11{,}000$ kg/year

Step 2: Allocate waste by mass fraction.

$W_A = (M_A / M_{total}) \times W_{total}$

$W_A = (5000 / 11000) \times 800 = 364$ kg/year

$W_B = (M_B / M_{total}) \times W_{total}$

$W_B = (6000 / 11000) \times 800 = 436$ kg/year

Step 3: Footprint per unit.

$PPF_A = (M_A + W_A) / 10000 = (5000 + 364) / 10000 = 0.536$ kg/unit

$PPF_B = (M_B + W_B) / 5000 = (6000 + 436) / 5000 = 1.287$ kg/unit

Answer: PPF_A = 536g/unit, PPF_B = 1,287g/unit.

Example 3: Functional Unit Comparison

Given: Two packaging options for 1kg of product: - Option A: 50g plastic, single-use - Option B: 200g plastic, reusable 10 times

Find: Footprint per use.

Solution:

Step 1: Calculate footprint per use for Option A.

$PPF_A = 50g / 1$ use $= 50$ g/use

Step 2: Calculate footprint per use for Option B.

$PPF_B = 200g / 10$ uses $= 20$ g/use

Step 3: Compare.

```
Reduction = (50 - 20) / 50 × 100% = 60%
```

Answer: Option B has 60% lower footprint per use (20g vs. 50g).

Example 4: Multi-Material Product

Given: A product contains: - Plastic: 300g - Metal: 500g - Glass: 200g

Calculate plastic footprint as percentage of total mass.

Solution:

Step 1: Calculate total mass.

$M_{total} = 300 + 500 + 200 = 1000g$

Step 2: Calculate plastic fraction.

$f_{plastic} = 300 / 1000 = 0.30 = 30\%$

Step 3: Plastic footprint.

```
PPF = 300g
```

Answer: Plastic footprint = 300g (30% of total product mass).

Example 5: Organizational Footprint Aggregation

Given: A company sells: - Product line A: 100,000 units/year, PPF_A = 200g/unit - Product line B: 50,000 units/year, PPF_B = 500g/unit - Product line C: 200,000 units/year, PPF_C = 50g/unit

Find: Total organizational plastic footprint from sold products.

Solution:

Step 1: Calculate footprint from each product line.

$PF_A = 100000 \times 0.2 = 20{,}000$ kg

$PF_B = 50000 \times 0.5 = 25{,}000$ kg

$PF_C = 200000 \times 0.05 = 10{,}000$ kg

Step 2: Sum total footprint.

PF_{total} = 20000 + 25000 + 10000 = 55,000 kg = 55 tonnes/year

Step 3: Calculate weighted average PPF.

PPF_{avg} = 55000 / (100000 + 50000 + 200000) = 55000 / 350000 = 0.157 kg/unit = 157 g/unit

Answer: Total footprint = 55 tonnes/year, Average PPF = 157g/unit.

PRACTICE PROBLEMS

Problem 1: A smartphone contains 45g of plastic. Manufacturing yield is 92% (8% defect rate). Calculate the plastic footprint including manufacturing losses.

Problem 2: Compare two packaging systems: (A) 20g plastic, recycled content 30%, (B) 15g plastic, virgin material. Which has lower virgin plastic footprint?

Problem 3: A product has PPF = 500g. If the functional unit is "per 1000 hours of use" and product lifetime is 5000 hours, calculate normalized footprint.

Problem 4: Allocate 1,200 kg of shared manufacturing waste between three products with annual production: A (10,000 units), B (15,000 units), C (5,000 units). Use economic allocation if prices are: A ($10), B ($15), C ($20).

Problem 5: A company's total plastic footprint is 10,000 tonnes/year with revenue of $100 million. Calculate plastic intensity (kg/$1000 revenue). If revenue grows 20% with no footprint change, what is the new intensity?

Chapter 14: DATA QUALITY AND TRACEABILITY

14.1 Introduction

Data quality is critical for credible plastic accounting. This chapter covers data collection, validation, quality assessment using pedigree matrices, and traceability systems.

14.2 Data Quality Indicators

Definition 14.1 (Pedigree Matrix)

Five-point scale (1 = best, 5 = worst) for: 1. **Reliability:** Verified data (1) vs. estimates (5) 2. **Completeness:** Complete data (1) vs. gaps (5) 3. **Temporal correlation:** Same year (1) vs. >10 years old (5) 4. **Geographical correlation:** Same location (1) vs. different continent (5) 5. **Technological correlation:** Same technology (1) vs. different (5)

Theorem 14.1 (Uncertainty from Data Quality)

Uncertainty factor U_i for indicator i with score s_i:

$$U_i = 1 + (s_i - 1) \times 0.1$$

Total uncertainty:

$$U_{total} = \sqrt{(\Sigma_i U_i^2)}$$

WORKED EXAMPLES

Example 1: Pedigree Matrix Application

Given: Data for plastic consumption has pedigree scores: - Reliability: 2 (measured with some uncertainty) - Completeness: 1 (complete dataset) - Temporal: 2 (data from 2 years ago) - Geographical: 1 (same location) - Technological: 3 (similar but not identical process)

Find: Total uncertainty factor.

Solution:

Step 1: Calculate uncertainty for each indicator.

$U_{reliability} = 1 + (2-1) \times 0.1 = 1.1$

$U_{completeness} = 1 + (1-1) \times 0.1 = 1.0$

$U_{temporal} = 1 + (2-1) \times 0.1 = 1.1$

$U_{geographical} = 1 + (1-1) \times 0.1 = 1.0$

$U_{technological} = 1 + (3-1) \times 0.1 = 1.2$

Step 2: Calculate total uncertainty.

$U_{total} = \sqrt{(1.1^2 + 1.0^2 + 1.1^2 + 1.0^2 + 1.2^2)}$

$U_{total} = \sqrt{(1.21 + 1.00 + 1.21 + 1.00 + 1.44)}$

$U_{total} = \sqrt{5.86} = 2.42$

Step 3: Interpret: Data has ~142% uncertainty ($U_{total} - 1 = 1.42$).

Answer: Total uncertainty factor = 2.42 (±142% uncertainty).

Example 2: Data Gap Filling

Given: Plastic consumption data available for 8 of 10 facilities: - Facilities 1-8: Total = 5,000 tonnes/year - Facilities 9-10: No data

Average consumption per facility (1-8): 625 tonnes/year

Find: Estimate total consumption including Facilities 9-10.

Solution:

Step 1: Calculate average from known facilities.

```
Avg = 5000 / 8 = 625 tonnes/year per facility
```

Step 2: Estimate for missing facilities.

Est_9-10 = 2 × 625 = 1,250 tonnes/year

Step 3: Total estimate.

$Total_{est}$ = 5000 + 1250 = 6,250 tonnes/year

Step 4: Assign data quality score: Completeness = 3 (80% complete).

Answer: Estimated total = 6,250 tonnes/year (Completeness score = 3).

Example 3: Temporal Correlation Adjustment

Given: Plastic consumption data from 2020: 1,000 tonnes/year Current year: 2025 Industry growth rate: 5% per year

Find: Adjusted estimate for 2025.

Solution:

Step 1: Calculate years elapsed.

```
Δt = 2025 - 2020 = 5 years
```

Step 2: Apply growth rate.

$C_{2025} = C_{2020} \times (1 + g)^{\Delta t}$
$C_{2025} = 1000 \times (1.05)^5$
$C_{2025} = 1000 \times 1.276 = 1{,}276$ tonnes/year

Step 3: Assign temporal correlation score: 3 (3-6 years old).

Answer: Adjusted consumption = 1,276 tonnes/year (Temporal score = 3).

Example 4: Traceability Chain

Given: A product's plastic supply chain: - Tier 1 supplier: 100% traceable - Tier 2 supplier: 60% traceable - Tier 3 supplier: 20% traceable

Find: Overall traceability percentage.

Solution:

Step 1: Calculate cascading traceability.

$T_{overall} = T_1 \times T_2 \times T_3$
$T_{overall} = 1.00 \times 0.60 \times 0.20 = 0.12 = 12\%$

Step 2: Interpret: Only 12% of plastic is fully traceable to Tier 3.

Answer: Overall traceability = 12%.

Example 5: Verification Sampling

Given: A facility reports 500 tonnes/year plastic consumption. Auditor samples 10% of purchase records, finding 48 tonnes in the sample period (representing 10% of the year).

Find: Verify the reported consumption.

Solution:

Step 1: Extrapolate sample to annual.

C_{sample} = 48 tonnes / 0.10 = 480 tonnes/year

Step 2: Compare to reported value.

```
Difference = |500 - 480| = 20 tonnes/year
Relative error = 20 / 500 = 0.04 = 4%
```

Step 3: If tolerance is ±5%, the report is verified.

Answer: Verified (4% error, within ±5% tolerance).

PRACTICE PROBLEMS

Problem 1: Calculate the uncertainty factor for data with pedigree scores: Reliability=3, Completeness=2, Temporal=4, Geographical=2, Technological=3.

Problem 2: A company has data for 75% of its plastic consumption. The known 75% totals 3,000 tonnes/year. Estimate total consumption and assign a completeness score.

Problem 3: Design a data collection protocol for a multi-site organization to achieve pedigree scores ≤2 for all indicators.

Problem 4: If traceability at each supply chain tier is 80%, what is the overall traceability for a 4-tier supply chain?

Problem 5: An auditor samples 5% of transactions and finds a 10% discrepancy. If the company reports 10,000 tonnes/year, what is the estimated actual consumption? What additional verification steps would you recommend?

Chapter 15: LEAKAGE QUANTIFICATION AND HOTSPOT ANALYSIS

15.1 Introduction

Plastic leakage to the environment is a critical metric for environmental impact. This chapter provides methods for quantifying leakage at different scales and identifying hotspots for intervention.

15.2 Leakage Estimation Methods

Definition 15.1 (Leakage Rate)

$LR = W_{mismanaged} \times P_{leakage}$

where: - $W_{mismanaged}$ = mismanaged waste (kg/year) - $P_{leakage}$ = probability of environmental leakage (dimensionless)

Theorem 15.1 (Spatial Leakage Model)

For geographic region i:

$L_i = W_i \times (1 - \eta_{collection,i}) \times (1 - \eta_{containment,i})$

where: - W_i = waste generation in region i (kg/year) - $\eta_{collection}$ = collection efficiency (fraction) - $\eta_{containment}$ = containment efficiency (fraction)

WORKED EXAMPLES

Example 1: National Leakage Estimation

Given: A country generates 5 million tonnes/year plastic waste. - Collection rate: 85% - Of collected waste, 95% is properly contained (landfill/recycling)

Find: Annual plastic leakage.

Solution:

Step 1: Calculate uncollected waste.

$W_{uncollected} = W_{total} \times (1 - \eta_{collection})$

$W_{uncollected} = 5{,}000{,}000 \times (1 - 0.85) = 750{,}000$ tonnes/year

Step 2: Calculate leakage from collected waste.

$W_{collected} = 5{,}000{,}000 \times 0.85 = 4{,}250{,}000$ tonnes/year

$L_{collected} = W_{collected} \times (1 - \eta_{containment})$

$L_{collected} = 4{,}250{,}000 \times (1 - 0.95) = 212{,}500$ tonnes/year

Step 3: Total leakage.

$L_{total} = W_{uncollected} + L_{collected}$

$L_{total} = 750{,}000 + 212{,}500 = 962{,}500$ tonnes/year

Step 4: Leakage rate.

$LR = L_{total} / W_{total} = 962{,}500 / 5{,}000{,}000 = 0.193 = 19.3\%$

Answer: Annual leakage = 962,500 tonnes/year (19.3% of waste generated).

Example 2: Facility-Level Leakage

Given: A manufacturing facility handles 10,000 tonnes/year of plastic pellets. Industry benchmark: 0.3% pellet loss.

Find: Annual leakage and daily rate.

Solution:

Step 1: Calculate annual leakage.

$L = M_{handled} \times f_{loss}$

$L = 10{,}000 \times 0.003 = 30$ tonnes/year

Step 2: Calculate daily rate.

$L_{daily} = 30{,}000$ kg / 365 days = 82.2 kg/day

Answer: Leakage = 30 tonnes/year or 82.2 kg/day.

Example 3: Hotspot Identification

Given: Five regions with waste and leakage: - Region A: W=1000 t/yr, L=50 t/yr - Region B: W=500 t/yr, L=100 t/yr - Region C: W=2000 t/yr, L=80 t/yr - Region D: W=800 t/yr, L=160 t/yr - Region E: W=1500 t/yr, L=60 t/yr

Find: Rank regions by leakage rate and identify top hotspot.

Solution:

Step 1: Calculate leakage rates.

$LR_A = 50/1000 = 0.050 = 5.0\%$

$LR_B = 100/500 = 0.200 = 20.0\%$

$LR_C = 80/2000 = 0.040 = 4.0\%$

$LR_D = 160/800 = 0.200 = 20.0\%$

$LR_E = 60/1500 = 0.040 = 4.0\%$

Step 2: Rank by leakage rate.

```
1. Region B & D: 20.0% (tied)
2. Region A: 5.0%
3. Region C & E: 4.0% (tied)
```

Step 3: Consider absolute leakage for prioritization.

```
Region D: 160 t/yr (highest absolute)
Region B: 100 t/yr
```

Answer: Top hotspot: Region D (20% rate, 160 t/yr absolute leakage).

Example 4: Intervention Impact

Given: Current leakage: 1,000 tonnes/year Intervention: Improve collection from 70% to 85% Waste generation: 5,000 tonnes/year Containment efficiency: 90% (unchanged)

Find: Leakage reduction from intervention.

Solution:

Step 1: Calculate current leakage.

$L_{current} = W \times [(1-\eta_{coll}) + \eta_{coll} \times (1-\eta_{cont})]$
$L_{current} = 5000 \times [(1-0.70) + 0.70 \times (1-0.90)]$
$L_{current} = 5000 \times [0.30 + 0.07] = 5000 \times 0.37 = 1{,}850$ t/yr

Wait, given says current = 1,000 t/yr. Let me recalculate to match:

Step 1: Calculate leakage after intervention.

$L_{after} = W \times [(1-\eta_{coll\_new}) + \eta_{coll\_new} \times (1-\eta_{cont})]$
$L_{after} = 5000 \times [(1-0.85) + 0.85 \times (1-0.90)]$
$L_{after} = 5000 \times [0.15 + 0.085] = 5000 \times 0.235 = 1{,}175$ t/yr

Step 2: If current is 1,000 t/yr (given), recalculate reduction: Assume current leakage = 1,000 t/yr corresponds to 70% collection.

Step 3: Reduction.

Reduction = $L_{current} - L_{after} = 1{,}000 - 1{,}175$

This doesn't match. Let me use the formula consistently:

Current: $\eta = 0.70$

$L_{current} = 5000 \times [(0.30) + 0.70 \times (0.10)] = 5000 \times 0.37 = 1{,}850$ t/yr

After: $\eta = 0.85$

$L_{after} = 5000 \times [(0.15) + 0.85\times(0.10)] = 5000 \times 0.235 = 1{,}175$ t/yr

Reduction:

```
ΔL = 1,850 - 1,175 = 675 t/yr
Reduction % = 675/1850 = 36.5%
```

Answer: Leakage reduction = 675 tonnes/year (36.5% reduction).

Example 5: Uncertainty in Leakage Estimates

Given: Waste generation: $W = 10{,}000 \pm 1{,}000$ tonnes/year Collection efficiency: $\eta = 0.80 \pm 0.05$ Leakage probability of uncollected: $P = 0.50 \pm 0.10$

Find: Leakage with uncertainty.

Solution:

Step 1: Calculate mean leakage.

$L_{mean} = W \times (1 - \eta) \times P$

$L_{mean} = 10{,}000 \times (1 - 0.80) \times 0.50$

$L_{mean} = 10{,}000 \times 0.20 \times 0.50 = 1{,}000$ tonnes/year

Step 2: Apply uncertainty propagation (simplified).

$\sigma_L^2 \approx (\partial L/\partial W)^2 \sigma_W^2 + (\partial L/\partial \eta)^2 \sigma_\eta^2 + (\partial L/\partial P)^2 \sigma_P^2$

Step 3: Calculate partial derivatives.

```
∂L/∂W = (1−η) × P = 0.20 × 0.50 = 0.10
∂L/∂η = −W × P = −10,000 × 0.50 = −5,000
∂L/∂P = W × (1−η) = 10,000 × 0.20 = 2,000
```

Step 4: Calculate variance.

$\sigma_L^2 = (0.10)^2(1000)^2 + (-5000)^2(0.05)^2 + (2000)^2(0.10)^2$

$\sigma_L^2 = 10{,}000 + 62{,}500 + 40{,}000 = 112{,}500$

$\sigma L = 335$ tonnes/year

Answer: Leakage = 1,000 ± 335 tonnes/year (±33.5% uncertainty).

PRACTICE PROBLEMS

Problem 1: A city generates 200,000 tonnes/year of plastic waste with 75% collection rate. Of uncollected waste, 80% leaks to the environment. Of collected waste, 5% leaks from landfills. Calculate total leakage.

Problem 2: Three facilities have leakage: Facility A (100 t/yr, handling 10,000 t/yr), Facility B (50 t/yr, handling 2,000 t/yr), Facility C (200 t/yr, handling 50,000 t/yr). Rank by leakage rate and identify the hotspot.

Problem 3: If improving collection efficiency from 60% to 80% reduces leakage by 500 tonnes/year, and waste generation is 10,000 tonnes/year, calculate the leakage probability of uncollected waste.

Problem 4: Design a monitoring program to detect and quantify microplastic leakage from a wastewater treatment plant. What parameters would you measure?

Problem 5: A country wants to reduce plastic leakage by 50% over 10 years. Current leakage is 1 million tonnes/year. What annual reduction rate is required? (Use exponential decay: $L(t) = L_0 \exp(-rt)$)

PART IV: STANDARDS AND FRAMEWORKS

Chapter 16: MASS BALANCE IN CHEMICAL RECYCLING

16.1 Introduction

Chemical recycling converts plastic waste back to monomers, oligomers, or chemical feedstocks through processes like pyrolysis, gasification, or depolymerization. Mass balance accounting is essential for tracking material flows and verifying recycled content claims.

16.2 Mass Balance Principles

Theorem 16.1 (Chemical Recycling Mass Balance)

For a chemical recycling process:

$M_{input} = M_{output} + M_{residue} + M_{emissions} + M_{losses}$

All terms in kg, with: - M_{input} = plastic waste feedstock - M_{output} = recycled product (monomers, oils, etc.) - $M_{residue}$ = solid residues (char, ash) - $M_{emissions}$ = gaseous emissions - M_{losses} = unaccounted losses

Definition 16.1 (Mass Balance Approach)

Recycled content allocated based on:

$RC = (M_{recycled\_input}) / (M_{total\_input})$

This RC factor can be applied to all outputs proportionally (mass balance accounting).

16.3 Chain of Custody Models

1. **Identity Preserved:** Physical segregation maintained
2. **Segregation:** Certified material kept separate
3. **Mass Balance:** Accounting tracks certified content through mixing
4. **Book & Claim:** Credits traded separately

WORKED EXAMPLES

Example 1: Pyrolysis Mass Balance

Given: Pyrolysis plant processes 10,000 tonnes/year mixed plastic waste: - Pyrolysis oil output: 6,500 tonnes/year - Gas output: 2,000 tonnes/year - Char residue: 1,300 tonnes/year - Losses: 200 tonnes/year

Find: Verify mass balance closure.

Solution:

Step 1: Sum all outputs.

$$M_{output\_total} = M_{oil} + M_{gas} + M_{char} + M_{losses}$$
$$M_{output\_total} = 6500 + 2000 + 1300 + 200 = 10{,}000 \text{ tonnes/year}$$

Step 2: Compare to input.

$$M_{input} = 10{,}000 \text{ tonnes/year}$$

Step 3: Check balance.

$$M_{input} = M_{output\_total} = 10{,}000 \text{ tonnes/year } \checkmark$$

Step 4: Calculate mass balance closure.

$$\text{Closure} = (M_{output\_total} / M_{input}) \times 100\% = 100\%$$

Answer: Mass balance is closed (100% closure).

Example 2: Recycled Content Allocation

Given: A chemical recycling facility mixes: - Recycled plastic feedstock: 3,000 tonnes/year - Virgin plastic feedstock: 7,000 tonnes/year - Total output: 9,500 tonnes/year (5% losses)

Find: Recycled content of output using mass balance approach.

Solution:

Step 1: Calculate total input.

$M_{input} = 3000 + 7000 = 10{,}000$ tonnes/year

Step 2: Calculate recycled content fraction.

$RC_{input} = M_{recycled} / M_{input} = 3000 / 10000 = 0.30 = 30\%$

Step 3: Apply to output (mass balance approach).

$RC_{output} = RC_{input} = 30\%$

Step 4: Calculate recycled mass in output.

$M_{recycled\_output} = RC_{output} \times M_{output}$

$M_{recycled\_output} = 0.30 \times 9500 = 2{,}850$ tonnes/year

Answer: Output has 30% recycled content (2,850 tonnes/year recycled plastic).

Example 3: Multi-Stage Process

Given: Two-stage chemical recycling: - Stage 1 (depolymerization): 1,000 kg input → 800 kg monomers + 200 kg residue - Stage 2 (repolymerization): 800 kg monomers → 750 kg recycled polymer + 50 kg losses

Find: Overall mass balance and yield.

Solution:

Step 1: Stage 1 mass balance.

$M_{in,1} = M_{out,1} + M_{residue,1}$

$1000 = 800 + 200$ ✓

Step 2: Stage 2 mass balance.

$M_{in,2} = M_{out,2} + M_{losses,2}$

800 = 750 + 50 ✓

Step 3: Overall yield.

$Yield = M_{out,2} / M_{in,1} = 750 / 1000 = 0.75 = 75\%$

Step 4: Total losses.

$Losses_{total} = M_{residue,1} + M_{losses,2} = 200 + 50 = 250$ kg (25%)

Answer: Overall yield = 75%, Total losses = 25%.

Example 4: Uncertainty in Mass Balance

Given: Input measurement: 10,000 ± 200 kg Output measurement: 9,500 ± 150 kg Residue measurement: 400 ± 50 kg

Find: Mass balance closure with uncertainty.

Solution:

Step 1: Calculate nominal closure.

$Closure = (M_{output} + M_{residue}) / M_{input}$

Closure = (9500 + 400) / 10000 = 0.99 = 99%

Step 2: Calculate uncertainty in numerator.

$\sigma_{num}^2 = \sigma_{output}^2 + \sigma_{residue}^2 = 150^2 + 50^2 = 22{,}500 + 2{,}500 = 25{,}000$

$\sigma num = 158$ kg

Step 3: Calculate relative uncertainty in closure.

$\sigma_{rel}^2 = (\sigma num / (Moutput + Mresidue))^2 + (\sigma input / Minput)^2$

$\sigma_{rel}^2 = (158 / 9900)^2 + (200 / 10000)^2$

$\sigma_{rel}^2 = 0.000255 + 0.0004 = 0.000655$

$\sigma rel = 0.0256 = 2.56\%$

Step 4: Absolute uncertainty in closure.

$\sigma_{closure} = Closure \times \sigma_{rel} = 0.99 \times 0.0256 = 0.025$

Answer: Mass balance closure = 99% ± 2.5%.

Example 5: Book & Claim System

Given: A company purchases 500 tonnes of recycled content credits but uses 100% virgin plastic physically (5,000 tonnes/year).

Find: Can the company claim recycled content? How much?

Solution:

Step 1: Under Book & Claim, credits can be purchased separately from physical material.

Step 2: Calculate claimed recycled content.

$RC_{claimed} = Credits / Total_{plastic} = 500 / 5000 = 0.10 = 10\%$

Step 3: Physical recycled content.

$RC_{physical} = 0\%$

Step 4: Disclosure requirement: Must state “10% recycled content claim via Book & Claim system (no physical recycled content).”

Answer: Can claim 10% recycled content via Book & Claim, but must disclose no physical recycled content.

PRACTICE PROBLEMS

Problem 1: A gasification plant has input 5,000 kg, syngas output 3,500 kg, slag 1,200 kg. Calculate the unaccounted losses and mass balance closure percentage.

Problem 2: If a mass balance system mixes 40% recycled and 60% virgin feedstock, and the process has 10% losses, what is the recycled content of the output?

Problem 3: Compare the credibility of: (a) Identity Preserved, (b) Mass Balance, (c) Book & Claim for recycled content claims. Which provides highest traceability?

Problem 4: A facility claims 95% mass balance closure. If input is 10,000 tonnes/year, what is the maximum acceptable unaccounted loss?

Problem 5: Design a mass balance accounting system for a chemical recycling facility that processes mixed plastic waste. What measurements are required at minimum?

Chapter 17: CHAIN OF CUSTODY SYSTEMS

17.1 Introduction

Chain of custody (CoC) tracks plastic materials through supply chains to verify recycled content claims, ensure traceability, and prevent fraud.

17.2 CoC Models

Definition 17.1 (Chain of Custody Models)

1. **Identity Preserved (IP):** Physical segregation maintained throughout supply chain
2. **Segregation (SEG):** Certified material kept separate but may mix with same certification
3. **Mass Balance (MB):** Accounting system tracks certified content through mixing
4. **Book and Claim (B&C):** Credits traded separately from physical material

Theorem 17.1 (Mass Balance CoC)

For mixing process with n inputs:

$$RC_{output} = (\Sigma_i M_i \times RC_i) / (\Sigma_i M_i)$$

where M_i is mass and RC_i is recycled content of input i.

WORKED EXAMPLES

Example 1: Identity Preserved Tracking

Given: Batch of 100% recycled plastic (1,000 kg) tracked through: - Collection → Sorting → Recycling → Manufacturing → Product

No mixing with virgin material at any stage.

Find: Recycled content of final product.

Solution:

Step 1: Under Identity Preserved, material is physically segregated.

Step 2: No mixing occurs, so:

$RC_{product} = RC_{input} = 100\%$

Step 3: Traceability: Can trace back to specific collection batch.

Answer: Product has 100% recycled content with full traceability.

Example 2: Mass Balance Mixing

Given: A manufacturing facility receives: - Batch A: 500 kg, 100% recycled - Batch B: 1,500 kg, 0% recycled (virgin)

Batches are mixed in production.

Find: Recycled content of output using mass balance.

Solution:

Step 1: Calculate total mass.

$M_{total} = 500 + 1500 = 2{,}000$ kg

Step 2: Calculate weighted average recycled content.

$RC_{output} = (M_A \times RC_A + M_B \times RC_B) / M_{total}$
$RC_{output} = (500 \times 1.0 + 1500 \times 0) / 2000$
$RC_{output} = 500 / 2000 = 0.25 = 25\%$

Answer: Output has 25% recycled content (mass balance approach).

Example 3: Segregation Model

Given: A facility processes: - Certified recycled plastic: 2,000 kg/day - Virgin plastic: 3,000 kg/day

Products are labeled separately: “Recycled” vs. “Virgin”

Find: Can a product be labeled “100% recycled” under segregation model?

Solution:

Step 1: Under Segregation model, certified material is kept separate.

Step 2: Products made from the 2,000 kg recycled stream can be labeled “100% recycled.”

Step 3: Products from the 3,000 kg virgin stream are labeled “0% recycled.”

Step 4: No mass balance mixing allowed under segregation.

Answer: Yes, products from the recycled stream can be labeled “100% recycled” (up to 2,000 kg/day capacity).

Example 4: Book & Claim Credit Trading

Given: - Recycling facility produces 1,000 tonnes recycled plastic, sells 800 tonnes physically - Remaining 200 tonnes: Sells credits separately - Manufacturer A: Buys 200 tonnes virgin plastic + 200 tonnes credits

Find: What can Manufacturer A claim?

Solution:

Step 1: Manufacturer A physically uses 200 tonnes virgin plastic.

Step 2: Manufacturer A purchases 200 tonnes recycled content credits.

Step 3: Under Book & Claim:

$$RC_{claimed} = \text{Credits} / \text{Total} = 200 / 200 = 100\%$$

Step 4: Disclosure: Must state "100% recycled content via Book & Claim system."

Answer: Can claim 100% recycled content (via credits), but physical material is 100% virgin.

Example 5: Fraud Detection

Given: A supplier claims to deliver 500 tonnes of 100% recycled plastic annually. Audit finds: - Recycled feedstock purchases: 300 tonnes/year - Virgin feedstock purchases: 400 tonnes/year - Total sales: 600 tonnes/year (14% process losses)

Find: Is the recycled content claim credible?

Solution:

Step 1: Calculate maximum possible recycled output.

$$M_{recycled\_max} = M_{recycled\_input} \times (1 - loss_{rate})$$
$$M_{recycled\_max} = 300 \times (1 - 0.14) = 258 \text{ tonnes/year}$$

Step 2: Supplier claims 500 tonnes recycled, but can only produce 258 tonnes maximum.

Step 3: Discrepancy:

```
Overclaim = 500 - 258 = 242 tonnes/year
```

Step 4: Conclusion: Claim is not credible (fraud detected).

Answer: Claim is fraudulent; maximum credible recycled content = 258 tonnes/year (43% of claimed 600 tonnes output).

PRACTICE PROBLEMS

Problem 1: Compare the traceability and credibility of the four CoC models (IP, SEG, MB, B&C). Rank them from highest to lowest traceability.

Problem 2: A facility mixes three batches: (A) 1,000 kg at 100% recycled, (B) 2,000 kg at 50% recycled, (C) 1,000 kg at 0% recycled. Calculate the recycled content of the mixed output.

Problem 3: Under what circumstances is Book & Claim appropriate? When should it NOT be used?

Problem 4: Design an audit protocol to verify mass balance CoC claims. What records and measurements would you inspect?

Problem 5: A company wants to transition from Book & Claim to Mass Balance CoC. What systems and processes need to be implemented?

Chapter 18: ISO STANDARDS AND REPORTING FRAMEWORKS

18.1 Introduction

International standards provide harmonized methods for plastic accounting and reporting. This chapter covers key ISO standards and their application to plastic footprinting.

18.2 Relevant ISO Standards

ISO 14040/14044: Life Cycle Assessment

ISO 14046: Water Footprint (analogous structure for plastic)

ISO 14067: Carbon Footprint of Products

ISO 14021: Environmental Labels (recycled content claims)

ISO 14064-1: Organizational GHG accounting (template for plastic)

Theorem 18.1 (ISO Compliance Requirements)

For ISO-compliant plastic footprint:

1. Functional unit definition
2. System boundary specification
3. Data quality requirements
4. Allocation procedures
5. Uncertainty assessment
6. Critical review (for public claims)

WORKED EXAMPLES

Example 1: ISO 14040-Compliant LCA

Given: Product LCA for plastic bottle: - Functional unit: 1,000 liters of beverage delivered - System boundary: Cradle-to-grave - Impact category: Plastic footprint (kg plastic)

Find: Structure the LCA according to ISO 14040.

Solution:

Step 1: Goal and Scope Definition

```
Goal: Quantify plastic footprint of beverage delivery
Functional Unit: 1,000 liters delivered
System Boundary: Raw material extraction → manufacturing → use → disposal
```

Step 2: Life Cycle Inventory (LCI)

```
Bottle production: 30g × 1,000 = 30 kg
Caps: 3g × 1,000 = 3 kg
Labels: 2g × 1,000 = 2 kg
Transport packaging: 1g × 1,000 = 1 kg
Total: 36 kg plastic per 1,000 liters
```

Step 3: Life Cycle Impact Assessment (LCIA)

```
Plastic footprint = 36 kg / 1,000 L = 0.036 kg/L = 36 g/L
```

Step 4: Interpretation

```
Bottle is dominant contributor (83% of plastic footprint)
```

Answer: ISO 14040-compliant plastic footprint = 36 g/L.

Example 2: ISO 14021 Recycled Content Claim

Given: Product contains: - Pre-consumer recycled content: 20% - Post-consumer recycled content: 10% - Virgin content: 70%

Find: Proper ISO 14021-compliant claim.

Solution:

Step 1: ISO 14021 distinguishes pre-consumer vs. post-consumer.

Step 2: Total recycled content:

$RC_{total} = RC_{pre} + RC_{post} = 20\% + 10\% = 30\%$

Step 3: Compliant claim format:

```
"Contains 30% recycled content (20% pre-consumer, 10% post-consumer)"
```

Step 4: Must NOT claim "30% post-consumer" (would be misleading).

Answer: ISO 14021-compliant claim: "30% recycled content (20% pre-consumer, 10% post-consumer)".

Example 3: ISO 14064-1 Adaptation for Plastic

Given: Organizational plastic accounting following ISO 14064-1 structure: - Scope 1: 500 tonnes/year - Scope 2: 300 tonnes/year - Scope 3: 2,000 tonnes/year

Find: Report according to ISO 14064-1 principles.

Solution:

Step 1: Organizational boundary (ISO 14064-1 §5.2)

```
Operational control approach
```

Step 2: Quantification (ISO 14064-1 §5.3)

```
Scope 1 (Direct): 500 tonnes/year
Scope 2 (Indirect - Purchased): 300 tonnes/year
Scope 3 (Other Indirect): 2,000 tonnes/year
Total: 2,800 tonnes/year
```

Step 3: Data quality (ISO 14064-1 §5.4)

```
Pedigree matrix assessment for each scope
```

Step 4: Reporting (ISO 14064-1 §6)

```
Report must include: boundaries, methodologies, data quality, uncertainties
```

Answer: ISO 14064-1 adapted report: Total plastic footprint = 2,800 tonnes/year (Scope 1: 18%, Scope 2: 11%, Scope 3: 71%).

Example 4: Critical Review Requirement

Given: Company wants to make public claim: "50% reduction in plastic footprint." Baseline: 1,000 tonnes/year Current: 500 tonnes/year

Find: Is critical review required per ISO 14040?

Solution:

Step 1: ISO 14040 §6.3: Critical review required for: - Comparative assertions disclosed to public - LCA studies intended to support public claims

Step 2: This is a comparative assertion (50% reduction) for public disclosure.

Step 3: Critical review IS required.

Step 4: Review must be conducted by: - Internal expert (minimum) - External expert (recommended) - Panel of interested parties (for contentious claims)

Answer: Yes, critical review IS required per ISO 14040 §6.3 for public comparative claims.

Example 5: Allocation Procedure (ISO 14044)

Given: Co-production process produces: - Product A: 1,000 kg (market value $10,000) - Product B: 500 kg (market value $5,000)

Shared plastic input: 300 kg

Find: Allocate plastic to each product using ISO 14044 hierarchy.

Solution:

Step 1: ISO 14044 allocation hierarchy: 1. Avoid allocation (subdivision/system expansion) 2. Physical relationship 3. Other relationship (economic)

Step 2: If subdivision not possible, use physical allocation (mass):

$Allocation_A = (M_A / (M_A + M_B)) \times M_{plastic}$
$Allocation_A = (1000 / 1500) \times 300 = 200$ kg

$Allocation_B = (500 / 1500) \times 300 = 100$ kg

Step 3: Alternative: Economic allocation:

$Allocation_A = (V_A / (V_A + V_B)) \times M_{plastic}$
$Allocation_A = (10000 / 15000) \times 300 = 200$ kg

$Allocation_B = (5000 / 15000) \times 300 = 100$ kg

Step 4: In this case, both methods give same result (coincidence).

Answer: Product A: 200 kg plastic, Product B: 100 kg plastic (both physical and economic allocation).

PRACTICE PROBLEMS

Problem 1: Design an ISO 14040-compliant LCA study for a reusable vs. single-use plastic bag. Define functional unit, system boundary, and key impact categories.

Problem 2: A product claims "Made from 100% ocean plastic." What ISO 14021 requirements apply? What documentation is needed?

Problem 3: Adapt the GHG Protocol Scope 1/2/3 framework to plastic accounting following ISO 14064-1 principles. What are the key differences from carbon accounting?

Problem 4: When is critical review mandatory vs. optional under ISO 14040? Provide three scenarios for each.

Problem 5: Compare physical, economic, and exergetic allocation methods for a plastic recycling process that produces multiple grades of recycled resin. Which is most appropriate per ISO 14044?

PART V: CIRCULARITY AND OPTIMIZATION

Chapter 19: CIRCULARITY METRICS AND INDICATORS

19.1 Introduction

Circularity metrics quantify the degree to which plastic flows follow circular (vs. linear) pathways. This chapter derives and analyzes key circularity indicators.

19.2 Material Circularity Indicator (MCI)

Definition 19.1 (Material Circularity Indicator)

```
MCI = 1 - LFI = 1 - (V + W) / (2M)
```

where: - V = virgin material input (kg) - W = waste (unrecovered) (kg) - M = total material flow (kg) - LFI = Linear Flow Index

Theorem 19.1 (MCI Bounds)

$0 \leq MCI \leq 1$, where: - MCI = 0: fully linear (V = M, W = M) - MCI = 1: fully circular (V = 0, W = 0)

Proof:

Step 1: For physical system, $0 \leq V \leq M$ and $0 \leq W \leq M$.

Step 2: Therefore $0 \leq V + W \leq 2M$.

Step 3: Thus $0 \leq (V + W)/(2M) \leq 1$.

Step 4: Hence $0 \leq 1 - (V + W)/(2M) \leq 1$.

∎

WORKED EXAMPLES

Example 1: MCI Calculation

Given: A plastic system has: - Virgin input: V = 700 kg/year - Recycled input: R = 300 kg/year - Total consumption: M = 1,000 kg/year - Waste (unrecovered): W = 600 kg/year

Find: Material Circularity Indicator.

Solution:

Step 1: Verify mass balance.

```
M = V + R = 700 + 300 = 1,000 kg/year ✓
```

Step 2: Calculate MCI.

```
MCI = 1 − (V + W) / (2M)
MCI = 1 − (700 + 600) / (2 × 1000)
MCI = 1 − 1300 / 2000
MCI = 1 − 0.65 = 0.35
```

Answer: MCI = 0.35 (35% circular).

Example 2: Recycling Rate vs. MCI

Given: System A: Recycling rate RR = 30%, all recycled material reused System B: Recycling rate RR = 30%, but only 50% of recycled material actually reused

Find: Compare MCI for both systems.

Solution:

System A:

Assume M = 1,000 kg

R = 0.30 × 1,000 = 300 kg (all reused)

V = 1,000 - 300 = 700 kg

W = 1,000 - 300 = 700 kg (70% becomes waste)

$MCI_A = 1 - (700 + 700) / 2000 = 1 - 0.70 = 0.30$

System B:

$R_{collected} = 300$ kg

$R_{reused} = 0.50 \times 300 = 150$ kg

$V = 1{,}000 - 150 = 850$ kg

$W = 1{,}000 - 300 = 700$ kg (waste) + 150 kg (recycled but not reused) = 850 kg

$MCI_B = 1 - (850 + 850) / 2000 = 1 - 0.85 = 0.15$

Answer: $MCI_A = 0.30$, $MCI_B = 0.15$ **(System A is more circular despite same recycling rate).**

Example 3: Circularity Improvement Scenario

Given: Current: V = 800 kg, W = 700 kg, M = 1,000 kg, MCI = 0.25 Target: MCI = 0.50

Find: Required reduction in virgin input or waste.

Solution:

Step 1: Current MCI.

$MCI_{current} = 1 - (800 + 700) / 2000 = 0.25$ ✓

Step 2: Target equation.

$0.50 = 1 - (V_{new} + W_{new}) / 2000$

$(V_{new} + W_{new}) / 2000 = 0.50$

$V_{new} + W_{new} = 1{,}000$

Step 3: Current sum.

```
V + W = 800 + 700 = 1,500
```

Step 4: Required reduction.

```
Reduction = 1,500 - 1,000 = 500 kg
```

Step 5: Scenarios: - Reduce V by 500: V_{new} = 300, W = 700 - Reduce W by 500: V = 800, W_{new} = 200 - Reduce both by 250 each: V_{new} = 550, W_{new} = 450

Answer: Must reduce V + W by 500 kg total to achieve MCI = 0.50.

Example 4: Product-Level MCI

Given: A product contains: - Virgin plastic: 80g - Recycled plastic: 20g - Total: 100g

Product lifetime: 5 years, then 60% recycled, 40% waste.

Find: Product MCI.

Solution:

Step 1: Virgin input.

```
V = 80g
```

Step 2: Waste at end-of-life.

```
W = 0.40 × 100 = 40g
```

Step 3: Total material flow (for single product lifecycle).

```
M = 100g
```

Step 4: Calculate MCI.

```
MCI = 1 - (V + W) / (2M)
MCI = 1 - (80 + 40) / (2 × 100)
MCI = 1 - 120 / 200 = 1 - 0.60 = 0.40
```

Answer: Product MCI = 0.40.

Example 5: Organizational MCI Aggregation

Given: A company has three product lines: - Line A: M_A = 1,000 kg/yr, MCI_A = 0.20 - Line B: M_B = 2,000 kg/yr, MCI_B = 0.40 - Line C: M_C = 1,500 kg/yr, MCI_C = 0.30

Find: Weighted average organizational MCI.

Solution:

Step 1: Calculate total material flow.

$M_{total} = 1{,}000 + 2{,}000 + 1{,}500 = 4{,}500$ kg/yr

Step 2: Calculate weighted MCI.

$MCI_{org} = (M_A \times MCI_A + M_B \times MCI_B + M_C \times MCI_C) / M_{total}$
$MCI_{org} = (1000 \times 0.20 + 2000 \times 0.40 + 1500 \times 0.30) / 4500$
$MCI_{org} = (200 + 800 + 450) / 4500$
$MCI_{org} = 1450 / 4500 = 0.322$

Answer: Organizational MCI = 0.32 (32% circular).

PRACTICE PROBLEMS

Problem 1: Calculate MCI for a system with: V = 600 kg, R = 400 kg, M = 1,000 kg, recycling efficiency = 70% (30% losses in recycling).

Problem 2: Prove that MCI = 1 - (V + W)/(2M) is equivalent to MCI = (R - W)/(2M) + 0.5 where R is recycled input.

Problem 3: A company wants to improve MCI from 0.25 to 0.50 over 5 years. If current V + W = 1,500 kg/yr and M = 1,000 kg/yr, what annual reduction rate in V + W is required?

Problem 4: Compare MCI with simple recycling rate (RR = R/M). Under what conditions do they give similar vs. different results?

Problem 5: Design a circularity dashboard for a manufacturing company with 5 key metrics beyond MCI. Justify each metric's inclusion.

Chapter 20: OPTIMIZATION OF WASTE MANAGEMENT SYSTEMS

20.1 Introduction

Optimization methods identify cost-effective strategies for maximizing plastic recovery and minimizing environmental impact. This chapter covers linear programming, multi-objective optimization, and facility location problems.

20.2 Linear Programming Formulation

Theorem 20.1 (Waste Management Optimization)

Minimize total cost:

$\min \sum_i \sum_j c_{ij} x_{ij}$

Subject to:

$\sum_j x_{ij} = W_i$ (all waste from source i allocated)
$\sum_i x_{ij} \leq C_j$ (capacity constraint at facility j)
$x_{ij} \geq 0$ (non-negativity)

where: - x_{ij} = waste flow from source i to facility j (kg) - c_{ij} = unit cost ($/kg) - W_i = waste generation at source i (kg) - C_j = capacity of facility j (kg)

WORKED EXAMPLES

Example 1: Two-Source, Two-Facility Allocation

Given: Sources: S1 (100 tonnes), S2 (150 tonnes) Facilities: F1 (capacity 120 tonnes, cost $50/tonne), F2 (capacity 200 tonnes, cost $60/tonne) Transport costs: S1→F1 ($10/t), S1→F2 ($15/t), S2→F1 ($12/t), S2→F2 ($8/t)

Find: Optimal allocation to minimize total cost.

Solution:

Step 1: Define variables.

x_{11} = S1 → F1

x_{12} = S1 → F2

x_{21} = S2 → F1

x_{22} = S2 → F2

Step 2: Total costs (processing + transport).

$c_{11} = 50 + 10 = 60$ \$/t

$c_{12} = 60 + 15 = 75$ \$/t

$c_{21} = 50 + 12 = 62$ \$/t

$c_{22} = 60 + 8 = 68$ \$/t

Step 3: Formulate LP.

$\min\ 60x_{11} + 75x_{12} + 62x_{21} + 68x_{22}$

Subject to:

$x_{11} + x_{12} = 100$ (S1 waste)

$x_{21} + x_{22} = 150$ (S2 waste)

$x_{11} + x_{21} \leq 120$ (F1 capacity)

$x_{12} + x_{22} \leq 200$ (F2 capacity)

$x_{ij} \geq 0$

Step 4: Solve (greedy heuristic: allocate to lowest cost first).

Lowest cost: $c_{11} = 60$ → allocate max to x_{11}

$x_{11} = \min(100, 120) = 100$

Remaining S1: 0

Remaining F1 capacity: 120 - 100 = 20

Next lowest: $c_{21} = 62 \rightarrow$ allocate to x_{21}

$x_{21} = \min(150, 20) = 20$

Remaining S2: 150 - 20 = 130

Next: $c_{22} = 68 \rightarrow$ allocate to x_{22}

$x_{22} = 130$

Step 5: Calculate total cost.

```
Cost = 60(100) + 75(0) + 62(20) + 68(130)
Cost = 6,000 + 0 + 1,240 + 8,840 = $16,080
```

Answer: Optimal allocation: x_{11}=100, x_{21}=20, x_{22}=130, Total cost = $16,080.

Example 2: Multi-Objective Optimization

Given: Two objectives: 1. Minimize cost: C = $50,000 2. Minimize environmental impact: E = 100 tonnes CO2e

Two scenarios: - Scenario A: C_A = $45,000, E_A = 120 tonnes CO2e - Scenario B: C_B = $55,000, E_B = 80 tonnes CO2e

Find: Pareto frontier and trade-off.

Solution:

Step 1: Check if scenarios are Pareto optimal.

Scenario A vs. Current:

$C_A < C$ (better), but $E_A > E$ (worse) → Pareto optimal

Scenario B vs. Current:

$C_B > C$ (worse), but $E_B < E$ (better) → Pareto optimal

Step 2: All three scenarios are on Pareto frontier.

Step 3: Calculate trade-off between A and B.

$\Delta C = C_B - C_A = 55{,}000 - 45{,}000 = \$10{,}000$

$\Delta E = E_B - E_A = 80 - 120 = -40$ tonnes CO_2e

Trade-off: $10,000 / 40 tonnes = $250 per tonne CO2e reduced

Answer: Pareto frontier includes all three scenarios; trade-off = $250/tonne CO2e.

Example 3: Facility Location Optimization

Given: Three potential recycling facility locations with fixed costs and service areas: - Location A: Fixed cost $500k, serves 50,000 tonnes/yr - Location B: Fixed cost $300k, serves 30,000 tonnes/yr - Location C: Fixed cost $400k, serves 40,000 tonnes/yr

Total waste to be served: 60,000 tonnes/yr

Find: Optimal facility selection.

Solution:

Step 1: Evaluate single-facility options.

```
A alone: Serves 50,000 < 60,000 (insufficient)
B alone: Serves 30,000 < 60,000 (insufficient)
C alone: Serves 40,000 < 60,000 (insufficient)
```

Step 2: Evaluate two-facility combinations.

```
A + B: Serves 80,000, Cost = $800k
A + C: Serves 90,000, Cost = $900k
B + C: Serves 70,000, Cost = $700k (sufficient, lowest cost)
```

Step 3: Optimal solution.

```
Select B and C: Total cost = $700k, Capacity = 70,000 tonnes/yr
```

Answer: Build facilities B and C for total cost of $700k.

Example 4: Sensitivity Analysis

Given: Optimal solution has x_{11} = 100 tonnes at cost c_{11} = $60/tonne. Total cost = $16,080.

Find: How much can c_{11} increase before solution changes?

Solution:

Step 1: Current allocation: $x_{11} = 100$ (full S1 waste).

Step 2: Alternative: Send S1 to F2 at c_{12} = $75/tonne.

Step 3: Solution changes when $c_{11} \geq c_{12}$.

$c_{11,max} = c_{12}$ = $75/tonne

Step 4: Allowable increase.

$\Delta c_{11} = 75 - 60$ = $15/tonne

Answer: c_{11} can increase by up to $15/tonne before solution changes.

Example 5: Dynamic Programming for Sequential Decisions

Given: Three-stage waste processing: - Stage 1: Collection (2 options: A=$10/t, B=$12/t) - Stage 2: Sorting (2 options: C=$15/t, D=$18/t) - Stage 3: Recycling (2 options: E=$25/t, F=$22/t)

Find: Minimum cost path.

Solution:

Step 1: Enumerate all paths.

```
A-C-E: 10 + 15 + 25 = $50/t
A-C-F: 10 + 15 + 22 = $47/t
A-D-E: 10 + 18 + 25 = $53/t
```

```
A-D-F: 10 + 18 + 22 = $50/t
B-C-E: 12 + 15 + 25 = $52/t
B-C-F: 12 + 15 + 22 = $49/t
B-D-E: 12 + 18 + 25 = $55/t
B-D-F: 12 + 18 + 22 = $52/t
```

Step 2: Minimum cost path.

```
A-C-F: $47/tonne
```

Answer: Optimal path: A → C → F, Total cost = $47/tonne.

PRACTICE PROBLEMS

Problem 1: Formulate a linear program for 3 waste sources and 4 recycling facilities with capacity constraints. Write out all constraints explicitly.

Problem 2: A waste management system has two objectives: minimize cost and maximize recycling rate. If current cost is $1M with 60% recycling, and an alternative costs $1.2M with 75% recycling, calculate the marginal cost per percentage point increase in recycling rate.

Problem 3: Use the simplex method (or software) to solve the LP from Example 1. Verify the solution matches the greedy heuristic result.

Problem 4: In a facility location problem, if fixed costs are F_i and variable costs are v_i per tonne, derive the break-even volume where two facilities have equal total cost.

Problem 5: Design a multi-period optimization model for waste management where facility capacities can be expanded over time. What additional constraints and variables are needed?

Chapter 21: MICROPLASTIC TRANSPORT MODELING

21.1 Introduction

Microplastics (< 5 mm) are transported through air, water, and soil via advection, diffusion, and settling. This chapter develops mathematical models for microplastic fate and transport.

21.2 Advection-Diffusion-Reaction Equation

Theorem 21.1 (Microplastic Transport)

Concentration C(x,t) of microplastics evolves according to:

$$\partial C/\partial t + \nabla\cdot(vC) = \nabla\cdot(D\nabla C) - kC + S$$

where: - v = velocity field (m/s) [advection] - D = diffusion coefficient (m^2/s) - k = degradation rate (s^{-1}) - S = source term ($kg/(m^3\cdot s)$)

Proof: Derived from conservation of mass with Fickian diffusion (standard in environmental engineering).

21.3 Settling Velocity

Theorem 21.2 (Stokes' Law for Microplastic Settling)

For spherical particle with diameter d:

$$v_s = (\rho_p - \rho_f)\ g\ d^2 / (18\ \mu)$$

where: - ρ_p = particle density (kg/m^3) - ρ_f = fluid density (kg/m^3) - g = gravitational acceleration (9.81 m/s^2) - μ = dynamic viscosity (Pa·s)

WORKED EXAMPLES

Example 1: 1D River Transport

Given: River with constant velocity $v = 0.5$ m/s, length $L = 10$ km. Microplastic source at $x = 0$: $S = 1$ kg/s. Diffusion coefficient: $D = 10$ m²/s. No degradation ($k = 0$).

Find: Steady-state concentration profile $C(x)$.

Solution:

Step 1: Steady-state 1D advection-diffusion equation.

```
v dC/dx = D d²C/dx² + S
```

Step 2: For point source at $x = 0$, solution is:

```
C(x) = (S / (v A)) exp(-v x / D)
```

where A is cross-sectional area.

Step 3: Assume $A = 100$ m² (river cross-section).

```
C(x) = (1 / (0.5 × 100)) exp(-0.5 x / 10)
C(x) = 0.02 exp(-0.05 x)  [kg/m³]
```

Step 4: Concentration at $x = 1$ km $= 1000$ m.

```
C(1000) = 0.02 exp(-0.05 × 1000) = 0.02 exp(-50) ≈ 0  [negligible]
```

Answer: $C(x) = 0.02 \exp(-0.05x)$ kg/m³; concentration decays rapidly downstream.

Example 2: Settling Velocity Calculation

Given: Microplastic particle: - Diameter: $d = 100$ μm $= 10^{-4}$ m - Density: $\rho_p = 1{,}050$ kg/m³ (polyethylene) - Water density: $\rho_f = 1{,}000$ kg/m³ - Water viscosity: $\mu = 10^{-3}$ Pa·s

Find: Settling velocity.

Solution:

Step 1: Apply Stokes' law.

$v_s = (\rho_p - \rho_f)\, g\, d^2 / (18\ \mu)$

Step 2: Substitute values.

$v_s = (1050 - 1000) \times 9.81 \times (10^{-4})^2 / (18 \times 10^{-3})$
$v_s = 50 \times 9.81 \times 10^{-8} / (18 \times 10^{-3})$
$v_s = 4.905 \times 10^{-6} / 0.018$
$v_s = 2.725 \times 10^{-4}$ m/s

Step 3: Convert to mm/s and m/day.

$v_s = 0.273$ mm/s
$v_s = 0.273 \times 10^{-3} \times 86400 = 23.6$ m/day

Answer: Settling velocity = 0.273 mm/s or 23.6 m/day.

Example 3: Residence Time in Water Column

Given: Water depth: H = 10 m Settling velocity: v_s = 20 m/day (from Example 2)

Find: Residence time before settling to bottom.

Solution:

Step 1: Residence time formula.

$\tau = H / v_s$

Step 2: Calculate.

```
τ = 10 m / 20 m/day = 0.5 days = 12 hours
```

Answer: Residence time = 12 hours before settling to sediment.

Example 4: Atmospheric Dispersion

Given: Microplastic emission from source: Q = 0.1 kg/s Wind speed: u = 5 m/s Atmospheric diffusion: D_y = 10 m²/s (lateral), D_z = 5 m²/s (vertical) Distance downwind: x = 1 km

Find: Concentration at ground level (z = 0) directly downwind (y = 0).

Solution:

Step 1: Gaussian plume model (simplified).

$C(x, y, z) = (Q / (2\pi u \sigma y \sigma z)) \exp(-y^2/(2\sigma_y^2)) \exp(-z^2/(2\sigma_z^2))$

Step 2: Estimate dispersion parameters.

$\sigma_y \approx \sqrt{}(2 D_y x / u) = \sqrt{}(2 \times 10 \times 1000 / 5) = \sqrt{4000} = 63.2$ m
$\sigma_z \approx \sqrt{}(2 D_z x / u) = \sqrt{}(2 \times 5 \times 1000 / 5) = \sqrt{2000} = 44.7$ m

Step 3: Calculate concentration at (x=1000, y=0, z=0).

```
C = (0.1 / (2π × 5 × 63.2 × 44.7)) exp(0) exp(0)
C = 0.1 / (2π × 5 × 2825)
C = 0.1 / 88,857 = 1.13 × 10⁻⁶ kg/m³ = 1.13 mg/m³
```

Answer: Ground-level concentration at 1 km downwind = 1.13 mg/m³.

Example 5: Degradation Half-Life in Environment

Given: Microplastic degradation follows first-order kinetics: dC/dt = -kC Measured concentrations: - t = 0: C_0 = 100 particles/L - t = 365 days: C = 95 particles/L

Find: Degradation rate constant k and half-life.

Solution:

Step 1: First-order decay solution.

```
C(t) = C₀ exp(-kt)
```

Step 2: Solve for k.

```
95 = 100 exp(-k × 365)
0.95 = exp(-365k)
ln(0.95) = -365k
k = -ln(0.95) / 365 = 0.0513 / 365 = 1.405 × 10⁻⁴ day⁻¹
```

Step 3: Calculate half-life.

```
t₁/₂ = ln(2) / k = 0.693 / (1.405 × 10⁻⁴) = 4,933 days = 13.5 years
```

Answer: $k = 1.4 \times 10^{-4}$ day^{-1}, Half-life = 13.5 years.

PRACTICE PROBLEMS

Problem 1: Solve the 1D steady-state advection-diffusion equation with constant source S, velocity v, and diffusion D. Derive the analytical solution C(x).

Problem 2: Calculate settling velocity for three microplastic types: (a) PET (ρ=1,380 kg/m³, d=50 μm), (b) PP (ρ=900 kg/m³, d=200 μm), (c) PVC (ρ=1,400 kg/m³, d=100 μm). Which settles fastest?

Problem 3: A lake has depth 50 m and microplastic concentration 10 particles/L at surface. If settling velocity is 10 m/day and no resuspension occurs, how long until surface concentration drops to 1 particle/L?

Problem 4: Derive the Gaussian plume equation from the 3D advection-diffusion equation with steady-state and constant wind assumptions.

Problem 5: Design a field sampling program to measure microplastic transport in a river. What spatial and temporal resolution is needed to calibrate an advection-diffusion model?

PART VI: ADVANCED TOPICS AND APPLICATIONS

Chapter 22: LIFE CYCLE ASSESSMENT INTEGRATION

22.1 Introduction

Life Cycle Assessment (LCA) evaluates environmental impacts across a product's lifecycle. This chapter integrates plastic accounting with LCA methodology following ISO 14040/14044.

22.2 LCA Framework

Definition 22.1 (LCA Phases)

1. **Goal and Scope Definition:** Functional unit, system boundary
2. **Life Cycle Inventory (LCI):** Data collection on inputs/outputs
3. **Life Cycle Impact Assessment (LCIA):** Environmental impact quantification
4. **Interpretation:** Results analysis and recommendations

Theorem 22.1 (Impact Assessment)

Total environmental impact for impact category k:

$I_k = \Sigma_i (M_i \times EF_{i,k})$

where: - M_i = mass of plastic type i (kg) - $EF_{i,k}$ = emission factor for plastic i in impact category k

---

WORKED EXAMPLES

Example 1: Comparative LCA - Virgin vs. Recycled

Given: Functional unit: 1 kg of plastic resin - Virgin PET: Energy = 80 MJ/kg, GHG = 3.5 kg CO2e/kg - Recycled PET: Energy = 30 MJ/kg, GHG = 1.5 kg CO2e/kg

Find: Environmental savings from using recycled PET.

Solution:

Step 1: Calculate energy savings.

```
ΔEnergy = 80 − 30 = 50 MJ/kg (62.5% reduction)
```

Step 2: Calculate GHG savings.

```
ΔGHG = 3.5 − 1.5 = 2.0 kg CO2e/kg (57% reduction)
```

Answer: Recycled PET saves 50 MJ/kg energy (62.5%) and 2.0 kg CO2e/kg GHG (57%).

Example 2: Multi-Impact Assessment

Given: Plastic packaging LCA with three impact categories: - Climate change: 5 kg CO2e - Plastic footprint: 0.5 kg plastic - Water use: 20 L

Normalization factors (per capita annual impact): - Climate change: 10,000 kg CO2e - Plastic footprint: 50 kg plastic - Water use: 50,000 L

Find: Normalized impacts.

Solution:

Step 1: Normalize climate change.

$I_{climate,norm} = 5 / 10{,}000 = 0.0005$ person-equivalents

Step 2: Normalize plastic footprint.

$I_{plastic,norm} = 0.5 / 50 = 0.01$ person-equivalents

Step 3: Normalize water use.

$I_{water,norm} = 20 / 50{,}000 = 0.0004$ person-equivalents

Step 4: Identify hotspot.

```
Plastic footprint is dominant (0.01 >> 0.0005, 0.0004)
```

Answer: Plastic footprint is the dominant impact (0.01 person-eq), 20× larger than climate or water.

Example 3: Allocation in Multi-Output Process

Given: Recycling process produces: - High-grade recycled plastic: 600 kg (value $1,200) - Low-grade recycled plastic: 400 kg (value $400)

Total environmental burden: 1,000 kg CO2e

Find: Allocate burden using (a) mass, (b) economic value.

Solution:

Step 1: Mass allocation.

$Burden_{high}$ = (600 / 1000) × 1000 = 600 kg CO2e
$Burden_{low}$ = (400 / 1000) × 1000 = 400 kg CO2e

Step 2: Economic allocation.

Total value = $1,200 + $400 = $1,600

$Burden_{high}$ = (1200 / 1600) × 1000 = 750 kg CO2e
$Burden_{low}$ = (400 / 1600) × 1000 = 250 kg CO2e

Answer: (a) Mass: 600 & 400 kg CO2e, (b) Economic: 750 & 250 kg CO2e.

Example 4: Consequential LCA

Given: Policy intervention increases recycling rate from 30% to 50%. - Marginal recycled plastic displaces virgin plastic 1:1 - Virgin plastic: 80 MJ/kg - Recycled plastic: 30 MJ/kg - Total plastic consumption: 10,000 tonnes/year

Find: Consequential energy savings.

Solution:

Step 1: Calculate change in recycling.

```
ΔRecycling = (0.50 − 0.30) × 10,000 = 2,000 tonnes/year
```

Step 2: Consequential impact: 2,000 tonnes virgin displaced by recycled.

```
ΔEnergy = 2,000 × (80 − 30) = 2,000 × 50 = 100,000 MJ/year
```

Answer: Consequential energy savings = 100,000 MJ/year.

Example 5: Uncertainty Analysis in LCA

Given: Plastic footprint estimate: 100 ± 20 kg GHG emissions estimate: 500 ± 100 kg CO_2e

Correlation coefficient: $\rho = 0.6$

Find: Combined uncertainty if total impact = Plastic + GHG (weighted equally).

Solution:

Step 1: Calculate mean total impact.

$I_{mean} = 100 + 500 = 600$

Step 2: Calculate variance with correlation.

```
Var(I) = Var(Plastic) + Var(GHG) + 2ρ√(Var(Plastic)×Var(GHG))
Var(I) = 20² + 100² + 2(0.6)√(400×10,000)
Var(I) = 400 + 10,000 + 2(0.6)(2,000)
Var(I) = 400 + 10,000 + 2,400 = 12,800
```

Step 3: Standard deviation.

$\sigma_I = \sqrt{12{,}800} = 113$

Answer: Total impact = 600 ± 113 (19% uncertainty).

PRACTICE PROBLEMS

Problem 1: Conduct a cradle-to-grave LCA for a reusable plastic container used 50 times vs. 50 single-use containers. Define functional unit and system boundary.

Problem 2: If recycled plastic has 40% lower GHG emissions than virgin but requires 20% more water, how would you weight these impacts to make a decision?

Problem 3: Derive the relationship between attributional and consequential LCA for a marginal change in recycling rate. Under what conditions do they give the same result?

Problem 4: Calculate the uncertainty in LCA results if three input parameters have uncertainties of ±10%, ±20%, and ±15% (independent). Use error propagation.

Problem 5: Design a sensitivity analysis for a plastic product LCA with 10 input parameters. Which parameters should be prioritized for data quality improvement?

Chapter 23: INDUSTRIAL APPLICATIONS

23.1 Introduction

This chapter applies plastic accounting methods to key industrial sectors: manufacturing, packaging, automotive, electronics, and construction.

23.2 Sector-Specific Considerations

Manufacturing: - High production scrap rates (5-15%) - Closed-loop recycling opportunities - Process optimization potential

Packaging: - Short product lifetimes (days to months) - High recycling potential (if collected) - Extended producer responsibility (EPR)

Automotive: - Long product lifetimes (10-15 years) - Complex material mixtures - End-of-life vehicle (ELV) regulations

Electronics: - Hazardous additives (flame retardants) - Precious metal co-recovery - E-waste management challenges

Construction: - Very long lifetimes (50+ years) - Large in-use stock accumulation - Demolition waste management

---

WORKED EXAMPLES

Example 1: Manufacturing Scrap Optimization

Given: Injection molding process: - Production: 10,000 units/year - Plastic per unit: 500g - Scrap rate: 10%

Scrap can be reground and reused with 5% quality degradation per cycle.

Find: Virgin plastic requirement with and without scrap recycling.

Solution:

Step 1: Without recycling.

$M_{product}$ = 10,000 × 0.5 = 5,000 kg/year

M_{scrap} = 0.10 × M_{total}

M_{total} = $M_{product}$ / (1 - 0.10) = 5,000 / 0.90 = 5,556 kg/year

M_{virgin} = 5,556 kg/year

Step 2: With scrap recycling (assume 100% scrap reuse).

M_{scrap} = 0.10 × M_{total} = 0.10 × 5,556 = 556 kg/year

$M_{recycled}$ = 556 kg/year (reground scrap)

M_{virgin} = M_{total} - $M_{recycled}$ = 5,556 - 556 = 5,000 kg/year

Step 3: Virgin plastic savings.

```
Savings = 5,556 - 5,000 = 556 kg/year (10%)
```

Answer: With scrap recycling: Virgin plastic = 5,000 kg/year (10% reduction).

Example 2: Packaging EPR Calculation

Given: A company sells 1 million products/year with 50g packaging each. EPR fee: $0.10 per kg of packaging. Recycling rate: 60%.

Find: (a) Annual EPR fee, (b) Effective cost per unit.

Solution:

Step 1: Calculate total packaging.

$M_{packaging}$ = 1,000,000 × 0.050 = 50,000 kg/year

Step 2: Calculate EPR fee.

Fee = $M_{packaging}$ × Rate = 50,000 × 0.10 = $5,000/year

Step 3: Effective cost per unit.

$Cost_{per\_unit}$ = \$5,000 / 1,000,000 = \$0.005 = 0.5 cents/unit

Answer: (a) EPR fee = \$5,000/year, (b) Cost per unit = 0.5 cents.

Example 3: Automotive Plastic Lifecycle

Given: Vehicle contains 150 kg plastic. Lifetime: 12 years. End-of-life recycling rate: 70%.

Find: (a) Annual plastic flow for 1 million vehicles/year, (b) In-use stock.

Solution:

Step 1: Annual plastic in new vehicles.

M_{new} = 1,000,000 × 150 = 150,000,000 kg/year = 150,000 tonnes/year

Step 2: In-use stock (steady state).

S = M_{new} × Lifetime = 150,000 × 12 = 1,800,000 tonnes

Step 3: Annual EoL plastic.

M_{EoL} = M_{new} = 150,000 tonnes/year (steady state)

Step 4: Recycled plastic.

$M_{recycled}$ = 0.70 × 150,000 = 105,000 tonnes/year

Answer: (a) Annual flow = 150,000 tonnes/year, (b) In-use stock = 1.8 million tonnes.

Example 4: Electronics Takeback Program

Given: Electronics manufacturer sells 500,000 devices/year (200g plastic each). Takeback program collects 30% of EoL devices after 5-year lifetime.

Find: (a) Annual plastic in new devices, (b) Annual plastic collected.

Solution:

Step 1: Annual plastic in new devices.

M_{new} = 500,000 × 0.200 = 100,000 kg/year = 100 tonnes/year

Step 2: In steady state, EoL devices = new devices.

M_{EoL} = 100 tonnes/year

Step 3: Plastic collected via takeback.

$M_{collected}$ = 0.30 × 100 = 30 tonnes/year

Answer: (a) New devices = 100 tonnes/year, (b) Collected = 30 tonnes/year (30% takeback rate).

Example 5: Construction Plastic Stock

Given: Annual construction: 1 million m² floor area. Plastic intensity: 5 kg/m² (pipes, insulation, etc.). Building lifetime: 50 years.

Find: In-use stock after 50 years (steady state).

Solution:

Step 1: Annual plastic in new construction.

M_{new} = 1,000,000 × 5 = 5,000,000 kg/year = 5,000 tonnes/year

Step 2: Steady-state stock.

$S = M_{new} \times \text{Lifetime}$ = 5,000 × 50 = 250,000 tonnes

Answer: In-use stock = 250,000 tonnes (steady state).

PRACTICE PROBLEMS

Problem 1: A manufacturing facility has 12% scrap rate. If 80% of scrap is recycled internally, calculate the effective virgin plastic requirement per kg of product.

Problem 2: Compare EPR systems: (a) Fee per kg packaging, (b) Deposit-return scheme. Which provides stronger incentive for reduction? Quantify for a 500g package with $0.10/kg fee vs. $0.20 deposit.

Problem 3: An automotive OEM wants to increase recycled content from 10% to 30% in vehicles (150 kg plastic per vehicle, 1M vehicles/year). Calculate the required recycled plastic supply and potential virgin plastic displacement.

Problem 4: Design a takeback program for electronics to achieve 50% collection rate. What logistical, economic, and regulatory factors must be considered?

Problem 5: Estimate the plastic stock in a city's building infrastructure. Assume: 10 million m^2 floor area, 50-year average building age, 5 kg plastic/m^2. What happens to this stock over the next 20 years if construction rate doubles?

Chapter 24: CONSUMER GOODS AND RETAIL SECTORS

24.1 Introduction

Consumer goods and retail sectors are major plastic users, particularly for packaging. This chapter addresses accounting challenges specific to these sectors.

24.2 Retail Plastic Accounting

Definition 24.1 (Retail Plastic Footprint)

$PF_{retail} = PF_{packaging} + PF_{products} + PF_{operations}$

where: - $PF_{packaging}$ = plastic in product packaging sold - $PF_{products}$ = plastic content in products sold - $PF_{operations}$ = operational plastic (bags, wrapping, etc.)

Theorem 24.1 (Extended Producer Responsibility Allocation)

Under EPR, producer responsibility:

$R_{producer} = \alpha \times M_{packaging}$

where α is responsibility factor ($0 \leq \alpha \leq 1$, typically $\alpha = 1$ for packaging).

WORKED EXAMPLES

Example 1: Retail Footprint Calculation

Given: A supermarket chain sells: - Packaged goods: 10,000 tonnes product, 500 tonnes packaging - Plastic products: 200 tonnes - Operational plastic (bags, wrap): 50 tonnes

Find: Total retail plastic footprint.

Solution:

Step 1: Packaging footprint.

$PF_{packaging}$ = 500 tonnes/year

Step 2: Product footprint.

$PF_{products}$ = 200 tonnes/year

Step 3: Operational footprint.

$PF_{operations}$ = 50 tonnes/year

Step 4: Total footprint.

PF_{retail} = 500 + 200 + 50 = 750 tonnes/year

Answer: Total retail plastic footprint = 750 tonnes/year.

Example 2: E-Commerce Packaging Optimization

Given: Current: 100,000 shipments/year, 200g packaging per shipment. Optimized: Reduce packaging to 150g per shipment.

Find: Annual plastic reduction.

Solution:

Step 1: Current packaging.

$M_{current}$ = 100,000 × 0.200 = 20,000 kg/year

Step 2: Optimized packaging.

$M_{optimized}$ = 100,000 × 0.150 = 15,000 kg/year

Step 3: Reduction.

```
ΔM = 20,000 - 15,000 = 5,000 kg/year = 5 tonnes/year (25% reduction)
```

Answer: Packaging reduction = 5 tonnes/year (25%).

Example 3: Reusable Packaging System

Given: Single-use packaging: 100g per delivery, 1 use. Reusable packaging: 500g per container, 50 uses. Annual deliveries: 1 million.

Find: Plastic footprint comparison.

Solution:

Step 1: Single-use footprint.

PF_{single} = 1,000,000 × 0.100 = 100,000 kg/year

Step 2: Reusable footprint.

Containers needed = 1,000,000 / 50 = 20,000 (assuming steady state)
$PF_{reusable}$ = 20,000 × 0.500 / 1 year = 10,000 kg/year

Step 3: Reduction.

```
Reduction = (100,000 - 10,000) / 100,000 = 90%
```

Answer: Reusable system reduces footprint by 90% (100,000 → 10,000 kg/year).

Example 4: EPR Compliance

Given: A brand sells products with 1,000 tonnes/year packaging. EPR regulation requires 70% recycling rate. Current recycling: 50%.

Find: Additional recycling needed for compliance.

Solution:

Step 1: Current recycling.

$R_{current}$ = 0.50 × 1,000 = 500 tonnes/year

Step 2: Required recycling.

$R_{required} = 0.70 \times 1{,}000 = 700$ tonnes/year

Step 3: Additional recycling needed.

$\Delta R = 700 - 500 = 200$ tonnes/year

Answer: Must increase recycling by 200 tonnes/year to achieve EPR compliance.

Example 5: Consumer Behavior Modeling

Given: Reusable bag program: - Single-use bags: 10g each, 1 million bags/year - Reusable bags: 50g each, 100 uses - Adoption rate: 60% of customers

Find: Plastic reduction from program.

Solution:

Step 1: Bags displaced by reusables.

$Bags_{displaced} = 0.60 \times 1{,}000{,}000 = 600{,}000$ bags/year

Step 2: Plastic saved from single-use bags.

$M_{saved} = 600{,}000 \times 0.010 = 6{,}000$ kg/year

Step 3: Reusable bags needed.

$Reusable_{needed} = 600{,}000 / 100 = 6{,}000$ bags

Step 4: Plastic in reusable bags (amortized annually).

$M_{reusable} = 6{,}000 \times 0.050 / 1 = 300$ kg/year

Step 5: Net reduction.

$Net_{reduction} = 6{,}000 - 300 = 5{,}700$ kg/year = 5.7 tonnes/year (95% reduction)

Answer: Net plastic reduction = 5.7 tonnes/year (95%).

PRACTICE PROBLEMS

Problem 1: A retailer sells 500 product categories with varying packaging. Design a sampling strategy to estimate total packaging footprint within ±10% accuracy.

Problem 2: Compare three packaging reduction strategies: (a) Lightweighting (20% reduction), (b) Reusable systems (90% reduction, 40% adoption), (c) Elimination (100% reduction, 10% of products). Which achieves greatest absolute reduction?

Problem 3: Model the plastic footprint of a deposit-return scheme for beverage bottles. Assume: 1 billion bottles/year, 30g each, 80% return rate, 10 uses per bottle. Compare to single-use baseline.

Problem 4: A brand wants to achieve "plastic neutral" status by funding plastic collection equal to its footprint. If footprint is 10,000 tonnes/year and collection costs $200/tonne, calculate annual cost. Is this a credible sustainability strategy?

Problem 5: Design a consumer communication strategy for plastic reduction. How would you measure effectiveness? What metrics would you track?

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APPENDIX A: MATHEMATICAL REFERENCE

A.1 Common Distributions

Normal Distribution:

$f(x) = (1/(\sigma\sqrt{(2\pi)}))\ \exp(-(x-\mu)^2/(2\sigma^2))$

Mean: μ, Variance: σ^2

Lognormal Distribution:

$f(x) = (1/(x\sigma\sqrt{(2\pi)}))\ \exp(-(\ln x - \mu)^2/(2\sigma^2))$

Mean: $\exp(\mu + \sigma^2/2)$

Exponential Distribution:

$f(x) = \lambda \exp(-\lambda x)$

Mean: $1/\lambda$, Variance: $1/\lambda^2$

A.2 Matrix Operations

Matrix Inverse (2×2):

$A = [a\ b]$ $\quad A^{-1} = (1/\det(A))\ [\ d\ \ -b]$

$\quad [c\ d] \qquad\qquad\qquad [-c\ \ \ a]$

where $\det(A) = ad - bc$

Eigenvalues (2×2):

```
det(A - λI) = 0
λ = (tr(A) ± √(tr(A)² - 4det(A))) / 2
```

A.3 Differential Equations

First-Order Linear ODE:

$$dy/dt + p(t)y = q(t)$$

Solution: $y = \exp(-\int p\,dt)\,[\int q \exp(\int p\,dt)\,dt + C]$

Separation of Variables:

$$dy/dx = f(x)g(y)$$
$$\int dy/g(y) = \int f(x)dx$$

APPENDIX B: DATA SOURCES

B.1 Global Plastic Production Data

- Plastics Europe: Annual production statistics
- OECD Global Plastics Outlook
- UN Comtrade: International trade data

B.2 Waste Management Data

- World Bank: What a Waste database
- UNEP: Marine litter assessment
- National waste statistics (EPA, Eurostat, etc.)

B.3 Polymer Properties

- CAMPUS database (polymer properties)
- MatWeb (material property database)
- Scientific literature (degradation rates, etc.)

APPENDIX C: SOFTWARE TOOLS

C.1 Material Flow Analysis

- **STAN** (TU Wien)**:** MFA software
- **Umberto:** Industrial ecology modeling
- **OpenLCA:** Open-source LCA software

C.2 Optimization

- **GAMS:** General Algebraic Modeling System
- **Python scipy.optimize:** Optimization library
- **CPLEX/Gurobi:** Commercial solvers

C.3 Data Analysis

- **Python pandas:** Data manipulation
- **R:** Statistical analysis
- **MATLAB:** Numerical computing