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MATHEMATICAL NOTATION AND CONVENTIONS

Throughout this textbook, we employ rigorous mathematical notation to provide precise formulations of water accounting principles.

Vector and Tensor Notation

Scalars: Lowercase letters w or Greek letters α

Vectors: Bold lowercase letters w

Matrices: Bold uppercase letters W

Tensors: Calligraphic letters W

Einstein Summation Convention

Repeated indices imply summation:

$$w_i x_i = \sum_{i=1}^{n} w_i x_i$$

Key Operations

- **Inner product:** $w \cdot x = w_i x_i$
- **Hadamard (element-wise) product:** $w \circ x$
- **Tensor contraction:** $W_{ijk} v_k$ (sum over k)
- **Gradient:** $\nabla f = \frac{\partial f}{\partial x_i}$
- **Jacobian:** $J_{ij} = \frac{\partial f_i}{\partial x_j}$
- **Hessian:** $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$

Hydrological Notation

- P = Precipitation (mm or m³)
- Q = Runoff or discharge (mm or m³)
- ET = Evapotranspiration (mm or m³)

- ΔS = Change in storage (mm or m³)
- W = Water use or withdrawal (m³)
- WF = Water footprint (m³)
- WI = Water intensity (m³/unit)

Subscripts

- $\square_{blue}$ = Blue water (surface and groundwater)
- $\square_{green}$ = Green water (soil moisture from precipitation)
- $\square_{grey}$ = Grey water (pollution assimilation)
- $\square_{cons}$ = Consumptive use
- $\square_{withdrawal}$ = Water withdrawal
- $\square_{discharge}$ = Water discharge

Chapter 1: WATER BALANCE AND THE BUDYKO FRAMEWORK

1.1 THE WATER BALANCE EQUATION

The water balance equation is the fundamental principle of hydrology, expressing the conservation of mass for water in a defined system.

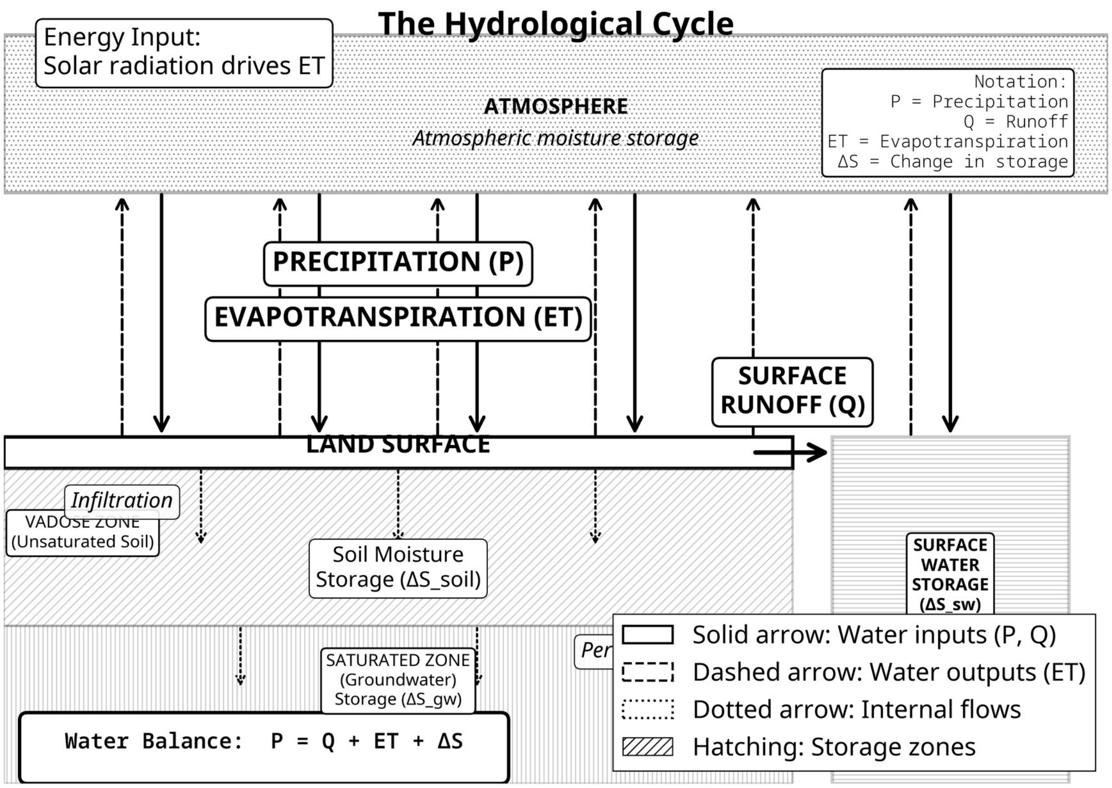

Figure 1.1: The Hydrological Cycle

Figure 1.1: The hydrological cycle showing the major water fluxes (precipitation, evapotranspiration, runoff) and storage components (soil moisture, groundwater, surface water). The water balance equation quantifies these flows.

Theorem 1.1 (Water Balance Conservation Law)

Statement:

For any hydrological system with defined boundaries over a time period Δt:

$$P+W_{¿}=Q+ET+\Delta S+W_{out}$$

where: - P = Precipitation (mm or m³) - $W_{¿}$ = Water inflow (imports, groundwater inflow) - Q = Runoff and discharge (mm or m³) - ET = Evapotranspiration (mm or m³) - ΔS = Change in storage (soil moisture, groundwater, surface water) - W_{out} = Water outflow (exports, groundwater outflow)

Proof:

The water balance is a direct application of the law of conservation of mass.

Step 1: Define the control volume.

Let $V(t)$ be the volume of water in the system at time t.

Step 2: Apply conservation of mass.

$$\frac{dV}{dt}=\text{Inflows}-\text{Outflows}$$

Step 3: Identify inflows and outflows.

Inflows: Precipitation P, surface water inflow $Q_{¿}$, groundwater inflow $GW_{¿}$

Outflows: Evapotranspiration ET, surface water outflow Q_{out}, groundwater outflow GW_{out}

Step 4: Integrate over time period Δt.

$$\int_{t_1}^{t_2}\frac{dV}{dt}\,dt=\int_{t_1}^{t_2}(¿P+Q_{¿}+GW_{¿}-ET-Q_{out}-GW_{out})\,dt¿$$

$$V(t_2)-V(t_1)=P+W_{¿}-ET-Q-W_{out}$$

$$\Delta S=P+W_{¿}-ET-Q-W_{out}$$

Rearranging:

$$P+W_{¿}=Q+ET+\Delta S+W_{out}$$

∎

Simplified Form (Closed Basin):

For a closed basin with no imports or exports:

$$P = Q + ET + \Delta S$$

Steady State:

At steady state ($\Delta S = 0$):

$$P = Q + ET$$

This is the fundamental equation for long-term water balance.

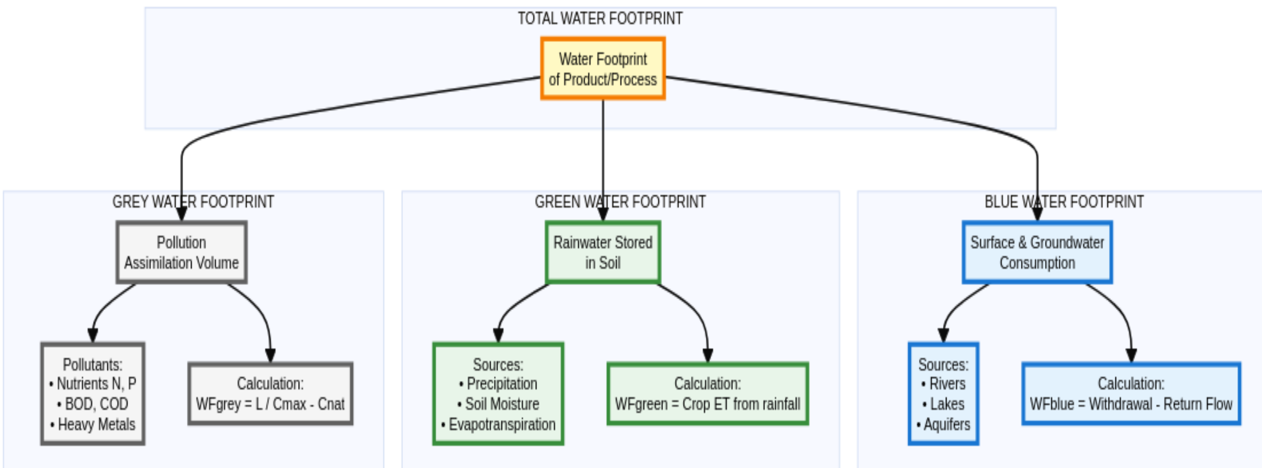

Figure 1.2: Blue-Green-Grey Water Components

Figure 1.2: Components of the water footprint: blue water (surface and groundwater), green water (soil moisture from precipitation), and grey water (pollution assimilation volume).

1.2 THE BUDYKO FRAMEWORK

The Budyko framework provides a theoretical foundation for understanding how climate controls the partitioning of precipitation into evapotranspiration and runoff at steady state.

1.2.1 Aridity Index and Evaporative Index

Aridity Index (Dryness Index):

$$\phi = \frac{E_0}{P}$$

where: - E_0 = Potential evapotranspiration (mm/year) - P = Mean annual precipitation (mm/year)

Evaporative Index:

$$\epsilon = \frac{ET}{P}$$

where ET is actual evapotranspiration.

Runoff Ratio:

$$\psi = \frac{Q}{P} = 1 - \epsilon$$

Theorem 1.2 (Budyko Hypothesis)

Statement:

For catchments at long-term steady state, the evaporative index ϵ is a function of the aridity index ϕ and a catchment parameter n:

$$\epsilon = f(\phi, n)$$

subject to the constraints: 1. $\epsilon \leq 1$ (cannot evaporate more than precipitation) 2. $\epsilon \leq \phi$ (cannot evaporate more than potential ET allows) 3. As $\phi \to 0$ (humid), $\epsilon \to \phi$ (energy-limited) 4. As $\phi \to \infty$ (arid), $\epsilon \to 1$ (water-limited)

Proof (Budyko Curve Derivation):

Step 1: Establish boundary conditions.

In extremely humid climates ($\phi \to 0$), water is abundant and evapotranspiration is limited by available energy:

$$ET \to E_0 \Rightarrow \epsilon \to \phi$$

In extremely arid climates ($\phi \to \infty$), energy is abundant and evapotranspiration is limited by available water:

$$ET \to P \Rightarrow \epsilon \to 1$$

Step 2: Propose functional form (Fu, 1981).

$$\epsilon = 1 + \phi - ¿$$

where n is a catchment parameter (typically 1 < n < 5).

Step 3: Verify boundary conditions.

As $\phi \to 0$:

$$\epsilon \approx 1 + \phi - ¿$$

More precisely, using Taylor expansion:

$$\epsilon \approx \phi - \frac{n-1}{2n}\phi^2 + O(\phi^3) \approx \phi$$

As $\phi \to \infty$:

$$\epsilon = 1 + \phi - \phi ¿$$

Step 4: Verify physical constraints.

For all $\phi > 0$ and $n > 1$, the Fu equation satisfies: - $0 \le \epsilon \le 1$ - $\epsilon \le \phi$ for $\phi \ge 1$

∎

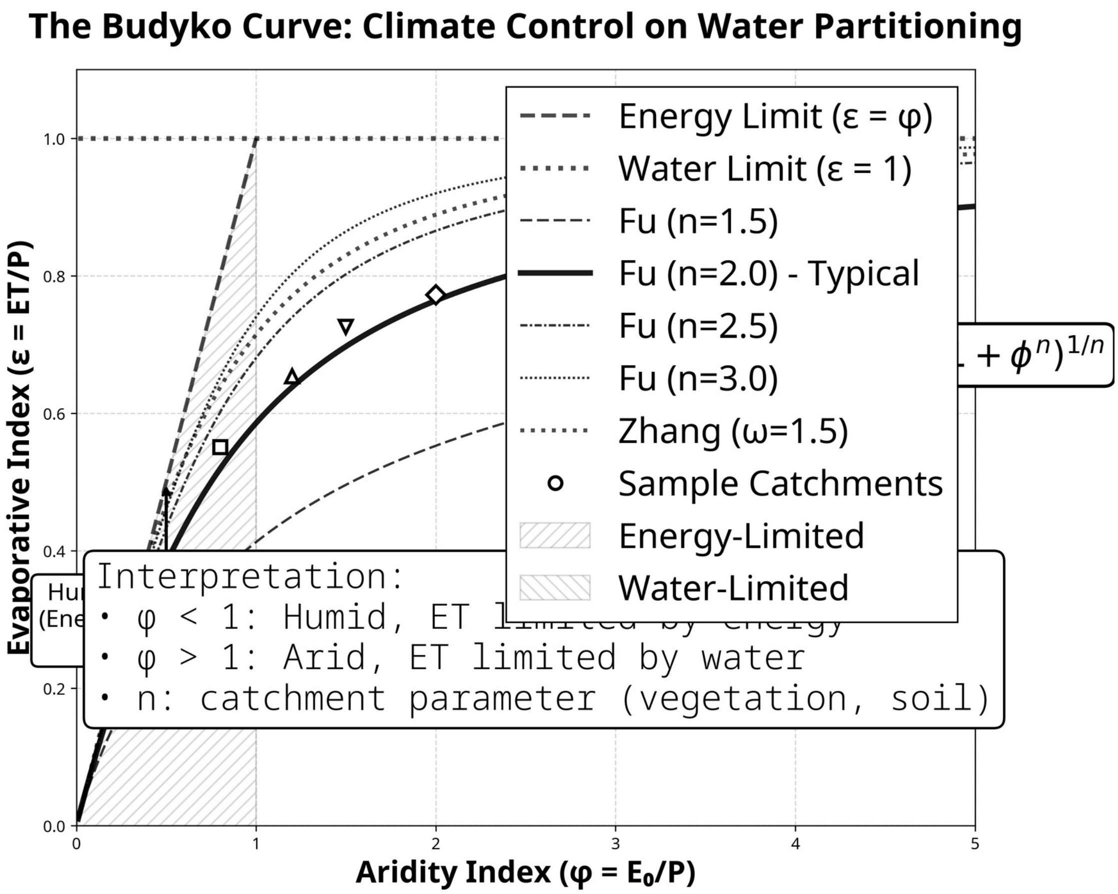

Figure 1.3: The Budyko Curve

Figure 1.3: The Budyko curve showing the relationship between aridity index (φ) and evaporative index (ϵ). The curve demonstrates how climate controls the partitioning of precipitation into evapotranspiration and runoff, with energy-limited conditions at low φ and water-limited conditions at high φ.

Alternative Formulation (Zhang et al., 2001):

$$\epsilon = \frac{1+\omega\phi}{1+\omega\phi+\phi^{-1}}$$

where ω is a plant-available water coefficient (typically $0.5 < \omega < 2.0$).

1.3 TEMPORAL AND SPATIAL SCALES

Water balance calculations depend critically on the temporal and spatial scales of analysis.

1.3.1 Temporal Scales

Annual Water Balance:

$$P_{annual} = Q_{annual} + ET_{annual} + \Delta S_{annual}$$

For multi-year averages, $\Delta S \approx 0$:

$$\acute{P} = \acute{Q} + \overline{ET}$$

Monthly Water Balance:

$$P_m = Q_m + ET_m + \Delta S_m$$

Storage changes are significant at monthly scale.

Daily Water Balance:

$$P_d = Q_d + ET_d + \Delta S_d$$

Storage changes dominate at daily scale.

1.3.2 Spatial Scales

Point Scale: Single measurement location (rain gauge, well)

Field Scale: Agricultural field or small catchment (< 1 km²)

Basin Scale: River basin or watershed (1-10,000 km²)

Regional Scale: Multiple basins (> 10,000 km²)

Global Scale: Continental or planetary water balance

SOLVED PROBLEMS

EXAMPLE 1.1

A catchment receives 1,200 mm annual precipitation. Streamflow is 400 mm/year. Assuming steady state ($\Delta S = 0$), calculate: (a) Evapotranspiration (b) Evaporative index (c) Runoff ratio

Solution:

(a) **Evapotranspiration:**

From water balance at steady state:

$$P = Q + ET$$

$$ET = P - Q = 1{,}200 - 400 = 800 \text{ mm/year}$$

Answer (a): 800 mm/year

(b) **Evaporative index:**

$$\epsilon = \frac{ET}{P} = \frac{800}{1{,}200} = 0.667$$

Answer (b): 0.667 (66.7% of precipitation evapotranspires)

(c) **Runoff ratio:**

$$\psi = \frac{Q}{P} = \frac{400}{1{,}200} = 0.333$$

Or: $\psi = 1 - \epsilon = 1 - 0.667 = 0.333$

Answer (c): 0.333 (33.3% of precipitation becomes runoff)

EXAMPLE 1.2

A basin has the following annual water balance (all in million m³): - Precipitation: 5,000 - Streamflow out: 1,500 - Groundwater inflow: 200 - Groundwater outflow: 300 - Change in storage: +100

Calculate evapotranspiration.

Solution:

Apply water balance equation:

$$P + W_{¿} = Q + ET + \Delta S + W_{out}$$

$$5{,}000 + 200 = 1{,}500 + ET + 100 + 300$$

$$5{,}200 = 1{,}900 + ET$$

$$ET = 5{,}200 - 1{,}900 = 3{,}300 \text{ million m}^3$$

Answer: 3,300 million m^3/year

Verification: - Inflows: 5,000 + 200 = 5,200 - Outflows: 1,500 + 3,300 + 100 + 300 = 5,200 ✓

EXAMPLE 1.3

Calculate the aridity index for a region with: - Mean annual precipitation: 800 mm - Potential evapotranspiration: 1,200 mm

Classify the climate.

Solution:

Aridity index:

$$\phi = \frac{E_0}{P} = \frac{1{,}200}{800} = 1.5$$

Answer: $\varphi = 1.5$

Classification: - $\varphi < 1$: Humid (energy-limited) - $\varphi = 1$: Transitional - $\varphi > 1$: Arid (water-limited)

This region is **arid** with water-limited evapotranspiration.

EXAMPLE 1.4

Using the Fu equation with n = 2, calculate the evaporative index for φ = 1.5.

Solution:

Fu equation:

$$\epsilon = 1 + \phi - \text{¿}$$

Substitute φ = 1.5, n = 2:

$$\epsilon = 1 + 1.5 - \text{¿}$$

$$\text{¿}\,2.5 - \text{¿}$$

$$\text{¿}\,2.5 - \text{¿}$$

$$\text{¿}\,2.5 - 1.803$$

$$\text{¿}\,0.697$$

Answer: ϵ = 0.697

Interpretation: 69.7% of precipitation evapotranspires, 30.3% becomes runoff.

EXAMPLE 1.5

A catchment has P = 1,000 mm/year and E_0 = 1,400 mm/year. Using the Zhang equation with ω = 1.5, calculate: (a) Evaporative index (b) Actual evapotranspiration (c) Runoff

Solution:

Step 1: Calculate aridity index:

$$\phi = \frac{E_0}{P} = \frac{1{,}400}{1{,}000} = 1.4$$

Step 2: Apply Zhang equation:

$$\epsilon = \frac{1+\omega\phi}{1+\omega\phi+\phi^{-1}}$$

$$¿\frac{1+1.5(1.4)}{1+1.5(1.4)+1.4^{-1}}$$

$$¿\frac{1+2.1}{1+2.1+0.714}$$

$$¿\frac{3.1}{3.814}=0.813$$

Answer (a): ϵ = 0.813

Step 3: Calculate actual ET:

$$ET=\epsilon \times P=0.813 \times 1{,}000=813 \text{ mm/year}$$

Answer (b): 813 mm/year

Step 4: Calculate runoff:

$$Q=P-ET=1{,}000-813=187 \text{ mm/year}$$

Answer (c): 187 mm/year

EXAMPLE 1.6

Prove that the runoff ratio ψ = 1 - ϵ.

Solution (Proof):

Given: Water balance at steady state:

$$P=Q+ET$$

Step 1: Divide both sides by P:

$$1=\frac{Q}{P}+\frac{ET}{P}$$

Step 2: Substitute definitions:

$$1 = \psi + \epsilon$$

Step 3: Rearrange:

$$\psi = 1 - \epsilon$$

∎

EXAMPLE 1.7

A reservoir has the following monthly water balance (all in m³): - Beginning storage: 10,000,000 - Precipitation on reservoir: 50,000 - Inflow from upstream: 2,000,000 - Evaporation: 150,000 - Outflow (release): 1,500,000 - Seepage loss: 20,000

Calculate ending storage.

Solution:

Apply water balance:

$$S_{end} = S_{begin} + P + Q_{\text{¿}} - E - Q_{out} - Seepage$$

$$S_{end} = 10{,}000{,}000 + 50{,}000 + 2{,}000{,}000 - 150{,}000 - 1{,}500{,}000 - 20{,}000$$

$$S_{end} = 10{,}380{,}000 \text{ m}^3$$

Change in storage:

$$\Delta S = 10{,}380{,}000 - 10{,}000{,}000 = 380{,}000 \text{ m}^3$$

Answer: Ending storage = 10,380,000 m³ (increase of 380,000 m³)

EXAMPLE 1.8

Derive the relationship between potential evapotranspiration and actual evapotranspiration in the Budyko framework.

Solution (Derivation):

From the definition of evaporative index:

$$\epsilon = \frac{ET}{P}$$

Therefore:

$$ET = \epsilon \times P$$

From the definition of aridity index:

$$\phi = \frac{E_0}{P}$$

Therefore:

$$P = \frac{E_0}{\phi}$$

Substituting:

$$ET = \epsilon \times \frac{E_0}{\phi}$$

For the Budyko curve, $\epsilon = f(\phi)$, so:

$$ET = f(\phi) \times \frac{E_0}{\phi}$$

Special cases:

When $\phi \to 0$ (humid): $\epsilon \to \phi$

$$ET \to \phi \times \frac{E_0}{\phi} = E_0$$

(Energy-limited: actual ET approaches potential ET)

When $\phi \to \infty$ (arid): $\epsilon \to 1$

$$ET \to 1 \times \frac{E_0}{\phi} = \frac{E_0}{\phi} = P$$

(Water-limited: actual ET approaches precipitation)

∎

EXAMPLE 1.9

A catchment has the following data over 5 years:

Year	P (mm)	Q (mm)
1	950	320
2	1,100	410
3	850	280
4	1,050	380
5	1,000	350

Calculate: (a) Mean annual precipitation (b) Mean annual runoff (c) Mean annual evapotranspiration (assuming ΔS = 0 over 5 years) (d) Mean evaporative index

Solution:

(a) **Mean annual precipitation:**

$$\acute{P}=\frac{950+1{,}100+850+1{,}050+1{,}000}{5}=\frac{4{,}950}{5}=990\ \text{mm}$$

Answer (a): 990 mm

(b) **Mean annual runoff:**

$$\acute{Q}=\frac{320+410+280+380+350}{5}=\frac{1{,}740}{5}=348\ \text{mm}$$

Answer (b): 348 mm

(c) **Mean annual ET:**

$$\overline{ET} = \acute{P} - \acute{Q} = 990 - 348 = 642 \text{ mm}$$

Answer (c): 642 mm

(d) **Mean evaporative index:**

$$\acute{\epsilon} = \frac{\overline{ET}}{\acute{P}} = \frac{642}{990} = 0.648$$

Answer (d): 0.648 (64.8%)

EXAMPLE 1.10

Show that the Fu equation satisfies the constraint ϵ ≤ 1 for all φ > 0 and n > 1.

Solution (Proof):

Fu equation:

$$\epsilon = 1 + \phi - ¿$$

Step 1: We need to show ¿ for all $\phi > 0$ and $n > 1$.

Step 2: Raise both sides to power n:

$$(1 + \phi^n) \geq \phi^n$$

Step 3: Simplify:

$$1 + \phi^n \geq \phi^n$$

$$1 \geq 0$$

This is always true.

Step 4: Therefore:

$$\epsilon = 1 + \phi - ¿$$

∎

Conclusion: The Fu equation always produces $\epsilon \leq 1$, satisfying the physical constraint that evapotranspiration cannot exceed precipitation at steady state.

SUPPLEMENTARY PROBLEMS

1.11 Catchment: P = 1,500 mm, Q = 600 mm, ΔS = 0. Find ET. **Ans.** 900 mm

1.12 Basin: P = 8,000 Mm³, Q = 2,500 Mm³, GW_{in} = 300 Mm³, GW_{out} = 400 Mm³, ΔS = +200 Mm³. Find ET. Ans. 5,200 Mm³

1.13 Region: P = 600 mm, E_0 = 1,500 mm. Find aridity index and classify. **Ans.** φ = 2.5 (arid)

1.14 Fu equation, n = 2.5, φ = 2.0. Find ϵ. **Ans.** 0.775

1.15 Zhang equation, ω = 1.2, φ = 1.8. Find ε. **Ans.** 0.796

1.16 Prove ψ + ϵ = 1 at steady state.

1.17 Reservoir: S_{begin} = 5M m³, P = 30k m³, Q_{in} = 1.5M m³, E = 80k m³, Q_{out} = 1.2M m³. Find S_{end}. Ans. 5.25M m³

1.18 Derive ET = ϵ × P from definitions.

1.19 3-year data: P = [800, 900, 850] mm, Q = [250, 300, 270] mm. Find mean ϵ. **Ans.** 0.682

1.20 Show Fu equation → φ as φ → 0 using L'Hôpital's rule.

Chapter 2: EVAPOTRANSPIRATION AND THE PENMAN-MONTEITH EQUATION

2.1 EVAPOTRANSPIRATION CONCEPTS

Evapotranspiration (ET): The combined process of water evaporation from soil and plant surfaces and transpiration through plant stomata.

Reference Evapotranspiration (ET_0): The evapotranspiration rate from a hypothetical reference crop (grass) with specific characteristics, used as a standard for comparison.

Crop Evapotranspiration (ETc): The evapotranspiration from disease-free, well-fertilized crops, grown in large fields, under optimum soil water conditions, and achieving full production under the given climatic conditions.

Potential Evapotranspiration (PET or E_0): The maximum evapotranspiration that would occur from a surface with unlimited water supply.

Actual Evapotranspiration (AET or ET): The actual evapotranspiration occurring under existing soil moisture and crop conditions.

2.2 THE PENMAN-MONTEITH EQUATION

Theorem 2.1 (Penman-Monteith Energy Balance)

Statement:

For a vegetated surface, the evapotranspiration rate can be calculated from energy balance and aerodynamic principles as:

$$\lambda ET = \frac{\Delta(R_n - G) + \rho_a c_p \frac{(e_s - e_a)}{r_a}}{\Delta + \gamma\left(1 + \frac{r_s}{r_a}\right)}$$

where: - λ = Latent heat of vaporization (MJ/kg) - ET = Evapotranspiration rate (kg/m²/s or mm/s) - Δ = Slope of saturation vapor pressure curve (kPa/°C) - R_n = Net radiation (MJ/m²/day) - G = Soil heat flux (MJ/m²/day) - ρ_a = Mean air density at constant pressure (kg/m³) - c_p = Specific heat of air (MJ/kg/°C) - e_s = Saturation vapor pressure (kPa) - e_a = Actual vapor pressure (kPa) - r_a = Aerodynamic resistance (s/m) - r_s = Surface (canopy) resistance (s/m) - γ = Psychrometric constant (kPa/°C)

Proof (Derivation):

Step 1: Energy balance at the surface.

$$R_n - G = H + \lambda ET$$

where H is sensible heat flux.

Step 2: Sensible heat flux (aerodynamic equation).

$$H = \rho_a c_p \frac{(T_s - T_a)}{r_a}$$

where T_s is surface temperature and T_a is air temperature.

Step 3: Latent heat flux (mass transfer).

$$\lambda ET = \rho_a c_p \frac{\gamma(e_s - e_a)}{r_a + r_s}$$

Step 4: Eliminate surface temperature using Clausius-Clapeyron relation.

$$e_s(T_s) - e_a = e_s(T_a) - e_a + \Delta(T_s - T_a)$$

Step 5: Substitute and rearrange.

From energy balance:

$$R_n - G = \rho_a c_p \frac{(T_s - T_a)}{r_a} + \lambda ET$$

Solving for $(T_s - T_a)$:

$$(T_s - T_a) = \frac{(R_n - G) r_a - \lambda ET \cdot r_a}{\rho_a c_p}$$

Step 6: Substitute into vapor pressure equation and solve for λET.

After algebraic manipulation:

$$\lambda ET = \frac{\Delta (R_n - G) + \rho_a c_p \frac{(e_s - e_a)}{r_a}}{\Delta + \gamma \left(1 + \frac{r_s}{r_a}\right)}$$

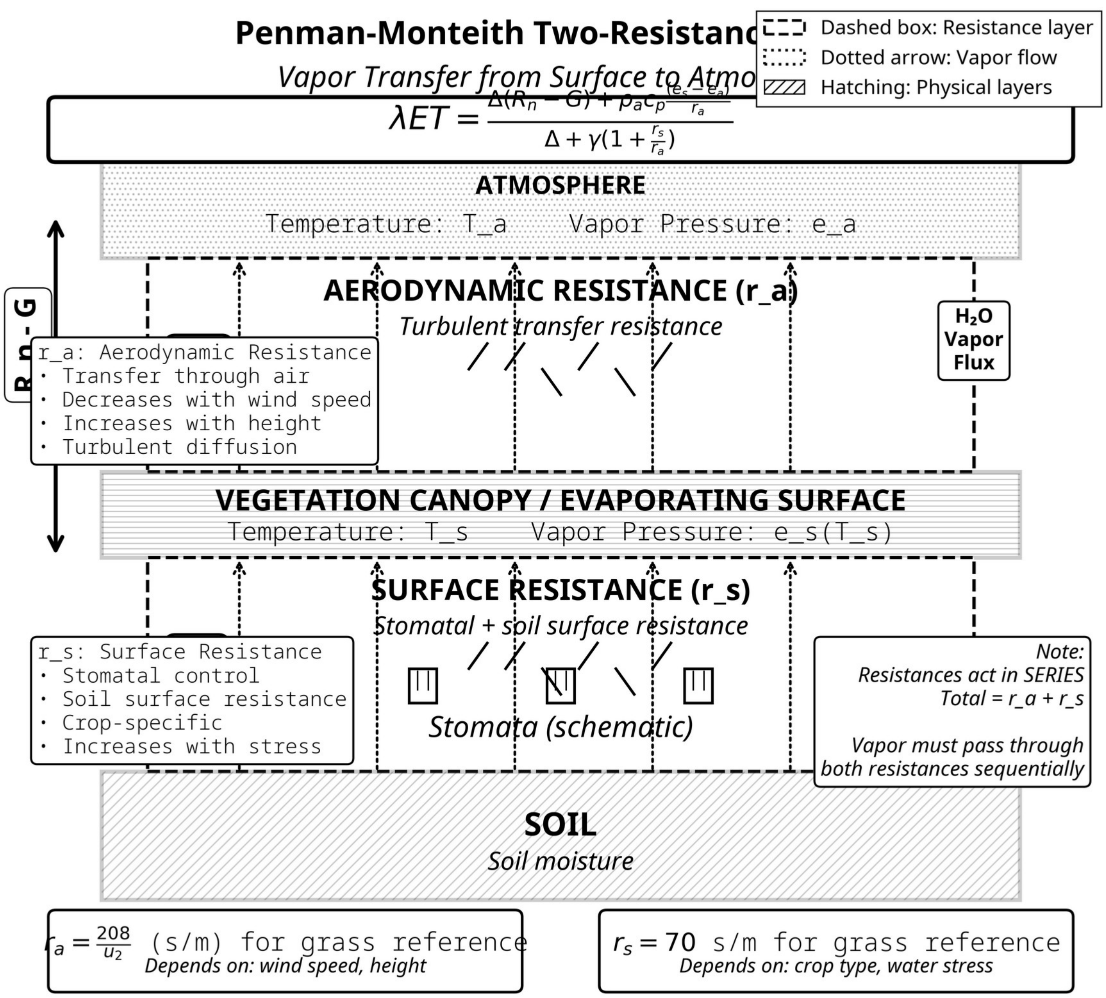

Figure 2.1: Penman-Monteith Two-Resistance Model

Figure 2.1: The Penman-Monteith two-resistance model showing aerodynamic resistance (ra) from the canopy to the atmosphere and surface resistance (rs) from stomatal and soil controls. Vapor flows from the vegetation surface through these resistances in series.

2.3 FAO PENMAN-MONTEITH EQUATION FOR REFERENCE ET

2.3.1 Reference Crop Definition

The FAO reference crop is defined as: - Hypothetical grass crop - Height: 0.12 m - Surface resistance: 70 s/m - Albedo: 0.23 - Actively growing, completely shading ground, well-watered

2.3.2 FAO Penman-Monteith Formula

For daily time step:

$$ET_0 = \frac{0.408\,\Delta(R_n - G) + \gamma \frac{900}{T+273} u_2 (e_s - e_a)}{\Delta + \gamma(1 + 0.34 u_2)}$$

where: - ET_0 = Reference evapotranspiration (mm/day) - R_n = Net radiation at crop surface (MJ/m²/day) - G = Soil heat flux density (MJ/m²/day) - T = Mean daily air temperature at 2 m height (°C) - u_2 = Wind speed at 2 m height (m/s) - e_s = Saturation vapor pressure (kPa) - e_a = Actual vapor pressure (kPa) - Δ = Slope of vapor pressure curve (kPa/°C) - γ = Psychrometric constant (kPa/°C)

Derivation of numerical constants:

The constant 0.408 converts MJ/m²/day to mm/day equivalent evaporation:

$$0.408 = \frac{1}{\lambda} \times 1000 = \frac{1}{2.45} \times 1000 \approx 408$$

The constant 900 arises from:

$$900 = \frac{86400 \times \rho_a c_p}{r_a \times \lambda}$$

where $r_a = 208/u_2$ for the reference surface.

The constant 0.34 arises from:

$$0.34 = \frac{r_s}{r_a} = \frac{70}{208/u_2} \times \frac{u_2}{1} = \frac{70u_2}{208} \approx 0.34\,u_2$$

2.4 CALCULATION OF PENMAN-MONTEITH PARAMETERS

2.4.1 Saturation Vapor Pressure

$$e_s(T) = 0.6108\exp\left(\frac{17.27\,T}{T + 237.3}\right)$$

where T is temperature in °C and e_s is in kPa.

For daily calculations:

$$e_s = \frac{e_s(T_{max}) + e_s(T_{min})}{2}$$

2.4.2 Actual Vapor Pressure

From relative humidity:

$$e_a = \frac{RH_{mean}}{100} \times e_s$$

From dewpoint temperature:

$$e_a = 0.6108\exp\left(\frac{17.27\,T_{dew}}{T_{dew} + 237.3}\right)$$

2.4.3 Slope of Vapor Pressure Curve

$$\Delta = \frac{4098 \times e_s(T)}{¿¿}$$

where T is mean air temperature in °C.

2.4.4 Psychrometric Constant

$$\gamma = \frac{c_p P}{\epsilon\lambda} = 0.665 \times 10^{-3}\,P$$

where: - P = Atmospheric pressure (kPa) - c_p = Specific heat at constant pressure = 1.013 kJ/kg/°C - ϵ = Ratio molecular weight water vapor/dry air = 0.622 - λ = Latent heat of vaporization = 2.45 MJ/kg

Pressure as function of elevation:

$$P = 101.3 \left(\frac{293 - 0.0065\, z}{293} \right)^{5.26}$$

where z is elevation above sea level (m).

2.4.5 Net Radiation

$$R_n = R_{ns} - R_{nl}$$

where: - R_{ns} = Net shortwave radiation (MJ/m²/day) - R_{nl} = Net longwave radiation (MJ/m²/day)

Net shortwave radiation:

$$R_{ns} = (1 - \alpha) R_s$$

where α = 0.23 (albedo for grass reference crop).

Net longwave radiation:

$$R_{nl} = \sigma \left[\frac{T_{max,K}^4 + T_{min,K}^4}{2} \right] \left(0.34 - 0.14 \sqrt{e_a} \right) \left(1.35 \frac{R_s}{R_{so}} - 0.35 \right)$$

where: - σ = Stefan-Boltzmann constant = 4.903×10^{-9} MJ/K⁴/m²/day - $T_{max,K}$, $T_{min,K}$ = Maximum and minimum absolute temperature (K) - R_s = Measured or calculated solar radiation (MJ/m²/day) - R_{so} = Clear-sky solar radiation (MJ/m²/day)

2.4.6 Soil Heat Flux

For daily calculations:

$$G \approx 0$$

(Soil heat flux is small compared to R_n for daily time steps)

For monthly calculations:

$$G = 0.07\left(T_{month,i+1} - T_{month,i-1}\right)$$

2.5 CROP EVAPOTRANSPIRATION

Theorem 2.2 (Crop Coefficient Method)

Statement:

Crop evapotranspiration can be calculated from reference ET using a crop coefficient:

$$ET_c = K_c \times ET_0$$

where: - ET_c = Crop evapotranspiration (mm/day) - K_c = Crop coefficient (dimensionless) - ET_0 = Reference evapotranspiration (mm/day)

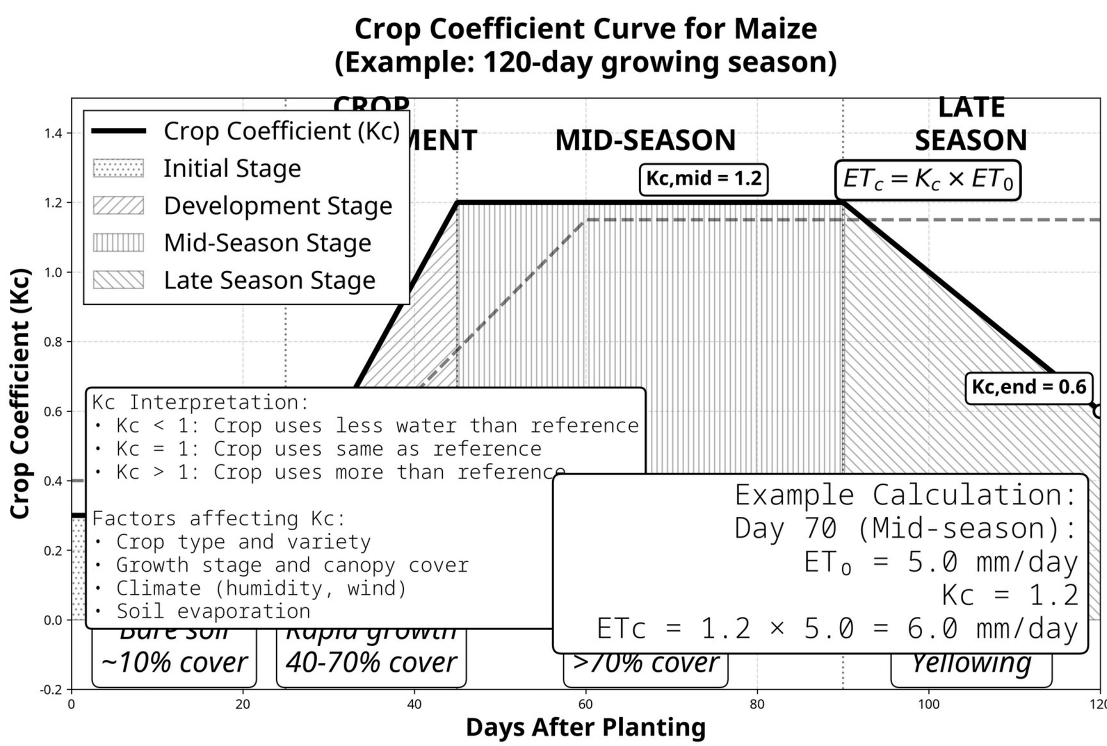

Figure 2.2: Crop Coefficient Curve

Figure 2.2: Crop coefficient (Kc) curve for maize showing variation through growth stages: initial (low canopy cover), development (rapid growth), mid-season (full canopy), and late season (senescence). The curve shape is typical for annual crops.

Proof (Justification):

The crop coefficient integrates the differences between the crop and reference surface:

Step 1: Ratio of evapotranspiration rates.

$$K_c = \frac{ET_c}{ET_0}$$

Step 2: Physical interpretation.

K_c accounts for: - Crop height (affects aerodynamic resistance) - Albedo (affects net radiation) - Canopy resistance (affects transpiration) - Soil evaporation

Step 3: Growth stage dependence.

K_c varies with crop development: - Initial stage: Low K_c (sparse canopy, high soil evaporation) - Mid-season: High K_c (full canopy, maximum transpiration) - Late season: Declining K_c (senescence, reduced transpiration)

∎

2.5.1 Crop Coefficient Curves

Single Crop Coefficient:

$$K_c = \begin{cases} K_{c,ini} & \text{Initial stage} \\ K_{c,mid} & \text{Mid-season} \\ K_{c,end} & \text{Late season} \end{cases}$$

with linear interpolation during development and late season stages.

Dual Crop Coefficient:

$$K_c = K_{cb} + K_e$$

where: - K_{cb} = Basal crop coefficient (transpiration only) - K_e = Soil evaporation coefficient

---

SOLVED PROBLEMS

EXAMPLE 2.1

Calculate saturation vapor pressure at T = 25°C.

Solution:

Using the equation:

$$e_s(T) = 0.6108 \exp\left(\frac{17.27\,T}{T+237.3}\right)$$

$$e_s(25) = 0.6108 \exp\left(\frac{17.27 \times 25}{25+237.3}\right)$$

$$¿0.6108 \exp\left(\frac{431.75}{262.3}\right)$$

$$¿0.6108 \exp(1.646)$$

$$¿0.6108 \times 5.185$$

$$¿3.167 \text{ kPa}$$

Answer: 3.167 kPa

EXAMPLE 2.2

Calculate actual vapor pressure from: - Mean temperature: 20°C - Relative humidity: 65%

Solution:

Step 1: Calculate saturation vapor pressure:

$$e_s(20) = 0.6108 \exp\left(\frac{17.27 \times 20}{20+237.3}\right)$$

$$¿0.6108\exp\left(\frac{345.4}{257.3}\right)$$

$$¿0.6108\exp(1.342)$$

$$¿0.6108\times 3.828$$

$$¿2.338\ \text{kPa}$$

Step 2: Calculate actual vapor pressure:

$$e_a=\frac{RH}{100}\times e_s=\frac{65}{100}\times 2.338=1.520\ \text{kPa}$$

Answer: e_a = 1.520 kPa

EXAMPLE 2.3

Calculate the slope of vapor pressure curve at T = 22°C.

Solution:

Step 1: Calculate $e_s(22)$:

$$e_s(22)=0.6108\exp\left(\frac{17.27\times 22}{22+237.3}\right)$$

$$¿0.6108\exp(1.466)=0.6108\times 4.333=2.647\ \text{kPa}$$

Step 2: Calculate slope:

$$\Delta=\frac{4098\times e_s(T)}{¿¿}$$

$$¿\frac{4098\times 2.647}{¿¿}$$

$$¿\frac{10{,}847}{67{,}284}=0.161\ \text{kPa/°C}$$

Answer: Δ = 0.161 kPa/°C

EXAMPLE 2.4

Calculate psychrometric constant at elevation z = 500 m.

Solution:

Step 1: Calculate atmospheric pressure:

$$P = 101.3\left(\frac{293 - 0.0065 \times 500}{293}\right)^{5.26}$$

$$¿\,101.3\left(\frac{293 - 3.25}{293}\right)^{5.26}$$

$$¿\,101.3\left(\frac{289.75}{293}\right)^{5.26}$$

$$¿\,101.3 \times ¿$$

$$¿\,101.3 \times 0.9423 = 95.46 \text{ kPa}$$

Step 2: Calculate psychrometric constant:

$$\gamma = 0.665 \times 10^{-3} \times P = 0.000665 \times 95.46 = 0.0635 \text{ kPa/°C}$$

Answer: γ = 0.0635 kPa/°C

EXAMPLE 2.5

Calculate reference ET_0 using FAO Penman-Monteith with the following data: - Tmax = 28°C, Tmin = 18°C - RHmean = 60% - u_2 = 2.5 m/s - Rn = 15 MJ/m²/day - G = 0 (daily calculation) - Elevation = 100 m

Solution:

Step 1: Calculate mean temperature:

$$T = \frac{28 + 18}{2} = 23\,°C$$

Step 2: Calculate saturation vapor pressures:

$$e_s(28) = 0.6108 \exp\left(\frac{17.27 \times 28}{28 + 237.3}\right) = 3.780 \text{ kPa}$$

$$e_s(18) = 0.6108 \exp\left(\frac{17.27 \times 18}{18 + 237.3}\right) = 2.064 \text{ kPa}$$

$$e_s = \frac{3.780 + 2.064}{2} = 2.922 \text{ kPa}$$

Step 3: Calculate actual vapor pressure:

$$e_a = \frac{60}{100} \times 2.922 = 1.753 \text{ kPa}$$

Step 4: Calculate slope of vapor pressure curve:

$$\Delta = \frac{4098 \times 2.922}{¿¿}$$

Step 5: Calculate psychrometric constant:

$$P = 101.3\left(\frac{293 - 0.65}{293}\right)^{5.26} = 100.1 \text{ kPa}$$

$$\gamma = 0.000665 \times 100.1 = 0.0666 \text{ kPa/°C}$$

Step 6: Apply FAO Penman-Monteith equation:

$$ET_0 = \frac{0.408 \times 0.177 \times (15 - 0) + 0.0666 \times \frac{900}{23 + 273} \times 2.5 \times (2.922 - 1.753)}{0.177 + 0.0666 \times (1 + 0.34 \times 2.5)}$$

Numerator:

$$¿ 0.408 \times 0.177 \times 15 + 0.0666 \times \frac{900}{296} \times 2.5 \times 1.169$$

$$¿ 1.083 + 0.0666 \times 3.041 \times 2.5 \times 1.169$$

$$¿ 1.083 + 0.592 = 1.675$$

Denominator:

$$¿ 0.177 + 0.0666 \times (1 + 0.85)$$

$$¿ 0.177 + 0.0666 \times 1.85 = 0.177 + 0.123 = 0.300$$

$$ET_0 = \frac{1.675}{0.300} = 5.58 \text{ mm/day}$$

Answer: ET_0 = 5.58 mm/day

EXAMPLE 2.6

Calculate crop evapotranspiration for maize at mid-season with Kc = 1.2 and ET_0 = 5.0 mm/day.

Solution:

$$ET_c = K_c \times ET_0 = 1.2 \times 5.0 = 6.0 \text{ mm/day}$$

Answer: ETc = 6.0 mm/day

Interpretation: Maize at full canopy transpires 20% more than the reference grass crop.

EXAMPLE 2.7

Derive the relationship between vapor pressure deficit (VPD) and relative humidity.

Solution (Derivation):

Step 1: Define VPD:

$$VPD = e_s - e_a$$

Step 2: Express e_a in terms of RH:

$$e_a = \frac{RH}{100} \times e_s$$

Step 3: Substitute:

$$VPD = e_s - \frac{RH}{100} \times e_s$$

$$\text{¿} e_s \left(1 - \frac{RH}{100}\right)$$

$$¿ e_s \times \frac{100-RH}{100}$$

■

Example: If e_s = 3.0 kPa and RH = 60%:

$$VPD = 3.0 \times \frac{40}{100} = 1.2 \text{ kPa}$$

EXAMPLE 2.8

Calculate seasonal crop water requirement for wheat with: - Growing season: 120 days - Kc,ini = 0.3 (30 days) - Kc,mid = 1.15 (60 days) - Kc,end = 0.4 (30 days) - Average ET_0 = 4.5 mm/day

Solution:

Initial stage:

$$CWR_{ini} = 0.3 \times 4.5 \times 30 = 40.5 \text{ mm}$$

Mid-season:

$$CWR_{mid} = 1.15 \times 4.5 \times 60 = 310.5 \text{ mm}$$

Late season:

$$CWR_{end} = 0.4 \times 4.5 \times 30 = 54.0 \text{ mm}$$

Total:

$$CWR_{total} = 40.5 + 310.5 + 54.0 = 405 \text{ mm}$$

Answer: 405 mm total crop water requirement

EXAMPLE 2.9

Show that the Penman-Monteith equation reduces to the Priestley-Taylor equation when aerodynamic term is negligible.

Solution (Proof):

Penman-Monteith equation:

$$\lambda ET = \frac{\Delta (R_n - G) + \rho_a c_p \frac{(e_s - e_a)}{r_a}}{\Delta + \gamma \left(1 + \frac{r_s}{r_a}\right)}$$

Step 1: Assume aerodynamic term is negligible:

$$\rho_a c_p \frac{(e_s - e_a)}{r_a} \approx 0$$

Step 2: Simplify:

$$\lambda ET = \frac{\Delta (R_n - G)}{\Delta + \gamma}$$

Step 3: Priestley-Taylor adds empirical coefficient α:

$$\lambda ET = \alpha \frac{\Delta (R_n - G)}{\Delta + \gamma}$$

where $\alpha \approx 1.26$ for well-watered surfaces.

∎

EXAMPLE 2.10

Calculate the water footprint of 1 kg of rice grain with: - Crop water requirement: 1,500 mm over growing season - Yield: 6,000 kg/ha - Field area: 1 ha

Solution:

Step 1: Convert CWR to volume:

$$V = 1{,}500 \text{ mm} \times 10{,}000 \text{ m}^2 \times 10^{-3} \text{ m/mm} = 15{,}000 \text{ m}^3/\text{ha}$$

Step 2: Calculate water intensity:

$$WI = \frac{15{,}000 \text{ m}^3}{6{,}000 \text{ kg}} = 2.5 \text{ m}^3/\text{kg}$$

Answer: 2.5 m^3/kg or 2,500 L/kg

SUPPLEMENTARY PROBLEMS

2.11 Calculate e_s at T = 30°C. **Ans.** 4.243 kPa

2.12 T = 18°C, RH = 70%. Find e_a. **Ans.** 1.445 kPa

2.13 T = 25°C. Calculate Δ. **Ans.** 0.189 kPa/°C

2.14 Elevation = 1,000 m. Find γ. **Ans.** 0.0601 kPa/°C

2.15 FAO PM: Tmax=30°C, Tmin=20°C, RH=55%, u_2=3 m/s, Rn=18 MJ/m²/day, z=200m. Find ET_0. **Ans.** 6.8 mm/day

2.16 Kc = 0.8, ET_0 = 4.2 mm/day. Find ETc. **Ans.** 3.36 mm/day

2.17 Derive VPD = e_s(1 - RH/100).

2.18 Season: 90 days, Kc,ini=0.4 (20d), Kc,mid=1.1 (50d), Kc,end=0.6 (20d), ET_0=5 mm/day. Find CWR. **Ans.** 415 mm

2.19 Show PM → Priestley-Taylor when aerodynamic term → 0.

2.20 CWR = 1,200 mm, yield = 8,000 kg/ha. Find WF per kg. **Ans.** 1.5 m³/kg

---

Chapter 3: WATER QUALITY AND POLLUTION ACCOUNTING

3.1 GREY WATER FOOTPRINT THEORY

Grey Water Footprint (WF_{grey}): The volume of freshwater required to assimilate a pollutant load based on natural background concentrations and existing ambient water quality standards.

Theorem 3.1 (Grey Water Footprint Assimilation)

Statement:

The grey water footprint for a pollutant is:

$$WF_{grey} = \frac{L}{C_{max} - C_{nat}}$$

where: - WF_{grey} = Grey water footprint (m³) - L = Pollutant load (kg or g) - C_{max} = Maximum acceptable concentration (kg/m³ or mg/L) - C_{nat} = Natural background concentration (kg/m³ or mg/L)

Proof (Derivation):

Step 1: Define assimilation requirement.

The pollutant must be diluted to meet water quality standards:

$$C_{effluent} \leq C_{max}$$

Step 2: Mass balance for dilution.

$$L = (C_{max} - C_{nat}) \times V_{dilution}$$

where $V_{dilution}$ is the volume of water needed for dilution.

Step 3: Solve for dilution volume.

$$V_{dilution} = \frac{L}{C_{max} - C_{nat}} = WF_{grey}$$

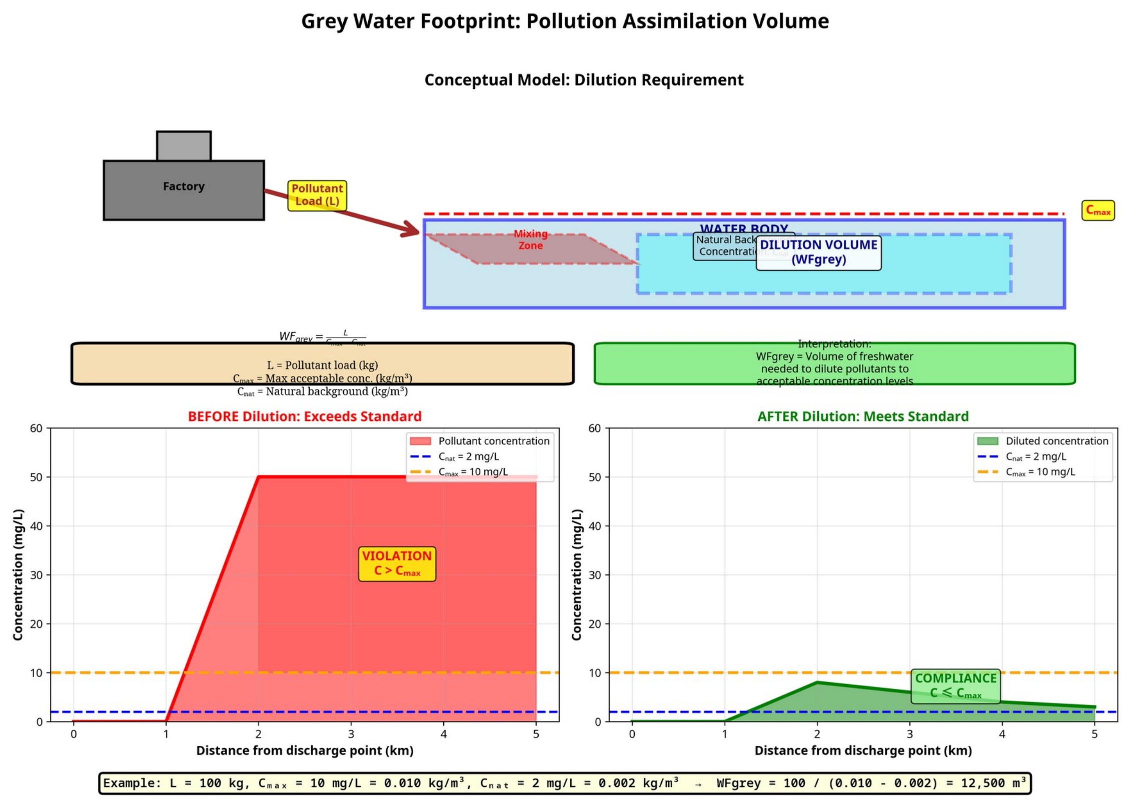

Figure 3.1: Grey Water Footprint Concept

Figure 3.1: The grey water footprint concept showing the dilution volume required to assimilate pollutants. The top panel illustrates the conceptual model with pollutant discharge, mixing zone, and dilution volume. The bottom panels show concentration profiles before and after dilution, demonstrating compliance with water quality standards.

Multi-Pollutant Case:

When multiple pollutants are present, the grey water footprint is the maximum across all pollutants:

$$WF_{grey,total} = \max_i \left(\frac{L_i}{C_{max,i} - C_{nat,i}} \right)$$

3.2 BIOCHEMICAL OXYGEN DEMAND (BOD)

BOD: The amount of dissolved oxygen required by aerobic microorganisms to decompose organic matter in water.

Theorem 3.2 (BOD Decay Kinetics)

Statement:

The BOD remaining at time t follows first-order decay kinetics:

$$BOD_t = BOD_0 \times e^{-k_1 t}$$

where: - BOD_t = BOD remaining at time t (mg/L) - BOD_0 = Initial BOD (mg/L) - k_1 = Deoxygenation rate constant (day^{-1}) - t = Time (days)

Proof (Derivation):

Step 1: Assume first-order reaction kinetics.

The rate of BOD removal is proportional to the BOD remaining:

$$\frac{dBOD}{dt} = -k_1 \times BOD$$

Step 2: Separate variables and integrate.

$$\frac{dBOD}{BOD} = -k_1 dt$$

$$\int_{BOD_0}^{BOD_t} \frac{dBOD}{BOD} = -k_1 \int_0^t dt$$

$$\ln(BOD_t) - \ln(BOD_0) = -k_1 t$$

Step 3: Solve for BOD_t.

$$\ln\left(\frac{BOD_t}{BOD_0}\right) = -k_1 t$$

$$BOD_t = BOD_0 \times e^{-k_1 t}$$

Ultimate BOD (BOD_u):

The total oxygen demand if decomposition proceeds to completion:

$$BOD_u = BOD_0 \times \frac{1}{1 - e^{-k_1 t}}$$

For standard 5-day BOD test:

$$BOD_5 = BOD_u \times \left(1 - e^{-5k_1}\right)$$

3.3 WATER QUALITY INDEX (WQI)

The Water Quality Index aggregates multiple water quality parameters into a single dimensionless number.

3.3.1 Weighted Arithmetic WQI

$$WQI = \frac{\sum_{i=1}^{n} W_i Q_i}{\sum_{i=1}^{n} W_i}$$

where: - W_i = Weight of parameter i - Q_i = Quality rating (sub-index) for parameter i (0-100)

3.3.2 Quality Rating Calculation

$$Q_i = \frac{V_i - V_{ideal}}{V_{standard} - V_{ideal}} \times 100$$

where: - V_i = Measured value of parameter i - V_{ideal} = Ideal value (often 0 for pollutants, 7 for pH) - $V_{standard}$ = Standard permissible value

3.3.3 WQI Classification

WQI Range	Water Quality	Suitability

0-25	Excellent	Drinking, all uses
26-50	Good	Drinking after treatment
51-75	Poor	Irrigation, industrial
76-100	Very Poor	Irrigation only
>100	Unsuitable	Restricted use

3.4 NUTRIENT POLLUTION ACCOUNTING

3.4.1 Nitrogen Species

Total Nitrogen (TN):

$$TN = NH_3\text{-}N + NO_2\text{-}N + NO_3\text{-}N + \text{Organic-}N$$

Nitrogen Load:

$$L_N = Q \times C_N \times t$$

where: - L_N = Nitrogen load (kg) - Q = Flow rate (m³/day) - C_N = Nitrogen concentration (kg/m³) - t = Time period (days)

3.4.2 Phosphorus Accounting

Total Phosphorus (TP):

$$TP = PO_4\text{-}P + \text{Organic-}P$$

Critical Load for Eutrophication:

For lakes and reservoirs:

$$L_{crit} = \frac{C_{crit} \times V}{\tau}$$

where: - L_{crit} = Critical phosphorus load (kg/year) - C_{crit} = Critical concentration (typically 0.01-0.02 mg/L) - V = Lake volume (m³) - τ = Hydraulic residence time (years)

3.4.3 Eutrophication Potential

Nutrient Ratio:

$$N:P=\frac{TN}{TP}$$

Classification: - N:P > 16: Phosphorus-limited - N:P < 16: Nitrogen-limited - N:P ≈ 16: Balanced (Redfield ratio)

3.5 POLLUTANT-SPECIFIC GREY WATER FOOTPRINTS

3.5.1 Common Pollutants and Standards

Pollutant	Symbol	Typical Standard (mg/L)
Biochemical Oxygen Demand	BOD_5	5-30
Chemical Oxygen Demand	COD	50-100
Total Suspended Solids	TSS	30-50
Total Nitrogen	TN	10-15
Total Phosphorus	TP	0.5-1.0
Heavy Metals (e.g., Pb)	Pb	0.01-0.05

3.5.2 Grey Water Footprint Calculation

For wastewater with multiple pollutants:

1. Calculate WF_{grey} for each pollutant
2. Take the maximum value (limiting pollutant)
3. Report both total and pollutant-specific values

SOLVED PROBLEMS

EXAMPLE 3.1

A factory discharges 100 kg of nitrogen into a river. The maximum acceptable concentration is 10 mg/L and natural background is 2 mg/L. Calculate the grey water footprint.

Solution:

Convert concentrations to kg/m³:

$$C_{max}=10 \text{ mg/L}=0.010 \text{ kg/m}^3$$

$$C_{nat}=2 \text{ mg/L}=0.002 \text{ kg/m}^3$$

Apply grey water footprint formula:

$$WF_{grey}=\frac{L}{C_{max}-C_{nat}}=\frac{100}{0.010-0.002}$$

$$\text{¿}\frac{100}{0.008}=12{,}500 \text{ m}^3$$

Answer: 12,500 m³ grey water footprint

EXAMPLE 3.2

BOD_0 = 200 mg/L, k_1 = 0.23 day^{-1}. Calculate: (a) BOD after 5 days (b) BOD after 10 days (c) Percentage removed after 5 days

Solution:

(a) **BOD after 5 days:**

$$BOD_5=BOD_0\times e^{-k_1 t}=200\times e^{-0.23\times 5}$$

$$\text{¿}200\times e^{-1.15}=200\times 0.317=63.4 \text{ mg/L}$$

Answer (a): 63.4 mg/L

(b) **BOD after 10 days:**

$$BOD_{10} = 200 \times e^{-0.23 \times 10} = 200 \times e^{-2.3}$$

$$= 200 \times 0.100 = 20.0 \text{ mg/L}$$

Answer (b): 20.0 mg/L

(c) **Percentage removed after 5 days:**

$$\% \text{ removed} = \frac{BOD_0 - BOD_5}{BOD_0} \times 100$$

$$= \frac{200 - 63.4}{200} \times 100 = \frac{136.6}{200} \times 100 = 68.3\%$$

Answer (c): 68.3% removed

EXAMPLE 3.3

Calculate WQI with the following parameters:

Parameter	Measured	Standard	Ideal	Weight
pH	7.5	8.5	7.0	4
DO (mg/L)	6.0	5.0	14.6	5
BOD (mg/L)	4.0	5.0	0	3
TDS (mg/L)	350	500	0	2

Solution:

Step 1: Calculate quality ratings.

pH:

$$Q_{pH} = \frac{|7.5 - 7.0|}{|8.5 - 7.0|} \times 100 = \frac{0.5}{1.5} \times 100 = 33.3$$

DO (higher is better, so invert):

$$Q_{DO} = \frac{6.0}{14.6} \times 100 = 41.1$$

BOD:

$$Q_{BOD} = \frac{4.0 - 0}{5.0 - 0} \times 100 = 80.0$$

TDS:

$$Q_{TDS} = \frac{350 - 0}{500 - 0} \times 100 = 70.0$$

Step 2: Calculate WQI:

$$WQI = \frac{4(33.3) + 5(41.1) + 3(80.0) + 2(70.0)}{4 + 5 + 3 + 2}$$

$$\text{¿} \frac{133.2 + 205.5 + 240.0 + 140.0}{14}$$

$$\text{¿} \frac{718.7}{14} = 51.3$$

Answer: WQI = 51.3 (Poor quality, suitable for irrigation/industrial use)

EXAMPLE 3.4

A wastewater stream has: - Flow: 1,000 m³/day - BOD: 150 mg/L - TN: 25 mg/L

Standards: BOD_{max} = 30 mg/L, TN_{max} = 10 mg/L (natural background = 0 for both)

Calculate: (a) Daily pollutant loads (b) Grey water footprint for each pollutant (c) Total grey water footprint

Solution:

(a) **Daily pollutant loads:**

BOD load:

$$L_{BOD} = 1{,}000 \times 0.150 = 150 \text{ kg/day}$$

TN load:

$$L_{TN} = 1{,}000 \times 0.025 = 25 \text{ kg/day}$$

Answer (a): BOD = 150 kg/day, TN = 25 kg/day

(b) **Grey water footprints:**

BOD:

$$WF_{grey,BOD} = \frac{150}{0.030 - 0} = 5{,}000 \text{ m}^3\text{/day}$$

TN:

$$WF_{grey,TN} = \frac{25}{0.010 - 0} = 2{,}500 \text{ m}^3\text{/day}$$

Answer (b): BOD = 5,000 m^3/day, TN = 2,500 m^3/day

(c) **Total grey water footprint:**

$$WF_{grey,total} = \max(5{,}000, 2{,}500) = 5{,}000 \text{ m}^3\text{/day}$$

Answer (c): 5,000 m^3/day (BOD is the limiting pollutant)

EXAMPLE 3.5

Derive the relationship between BOD_5 and ultimate BOD (BOD_u) for $k_1 = 0.1$ day^{-1}.

Solution (Derivation):

From BOD decay equation:

$$BOD_5 = BOD_u \times \left(1 - e^{-k_1 \times 5}\right)$$

For $k_1 = 0.1$ day^{-1}:

$$BOD_5 = BOD_u \times \left(1 - e^{-0.1 \times 5}\right)$$

$$¿ BOD_u \times (1 - e^{-0.5})$$

$$¿ BOD_u \times (1 - 0.607)$$

$$¿ BOD_u \times 0.393$$

Therefore:

$$BOD_u = \frac{BOD_5}{0.393} = 2.55 \times BOD_5$$

Answer: BOD_u ≈ 2.55 × BOD_5 for k_1 = 0.1 day^{-1}

EXAMPLE 3.6

A lake has volume V = 10^7 m^3 and hydraulic residence time τ = 2 years. Calculate the critical phosphorus load to prevent eutrophication (C_{crit} = 0.015 mg/L).

Solution:

Convert concentration:

$$C_{crit} = 0.015 \text{ mg/L} = 0.000015 \text{ kg/m}^3$$

Apply critical load formula:

$$L_{crit} = \frac{C_{crit} \times V}{\tau}$$

$$¿ \frac{0.000015 \times 10{,}000{,}000}{2}$$

$$¿ \frac{150}{2} = 75 \text{ kg/year}$$

Answer: 75 kg P/year critical load

Interpretation: Phosphorus inputs exceeding 75 kg/year will likely cause eutrophication.

EXAMPLE 3.7

A water sample has TN = 480 μg/L and TP = 30 μg/L. Determine if the system is nitrogen or phosphorus limited.

Solution:

Calculate N:P ratio (by mass):

$$N:P=\frac{TN}{TP}=\frac{480}{30}=16$$

Answer: N:P = 16 (Balanced, Redfield ratio)

Interpretation: The system is at the threshold between N and P limitation. Both nutrients should be managed.

EXAMPLE 3.8

Show that for two pollutants with the same load but different standards, the one with the stricter standard (lower C_{max}) dominates the grey water footprint.

Solution (Proof):

Given: - Pollutant A: Load = L, $C_{max,A} = C_A$ - Pollutant B: Load = L, $C_{max,B} = C_B$ - Assume $C_A < C_B$ (A has stricter standard) - Assume $C_{nat} = 0$ for simplicity

Step 1: Calculate grey water footprints:

$$WF_{grey,A}=\frac{L}{C_A}$$

$$WF_{grey,B}=\frac{L}{C_B}$$

Step 2: Compare:

Since $C_A < C_B$:

$$\frac{1}{C_A} > \frac{1}{C_B}$$

Therefore:

$$WF_{grey,A} > WF_{grey,B}$$

Step 3: Total grey water footprint:

$$WF_{grey,total} = \max(WF_{grey,A}, WF_{grey,B}) = WF_{grey,A}$$

∎

Conclusion: The pollutant with the stricter standard determines the grey water footprint.

EXAMPLE 3.9

A textile factory discharges wastewater with: - Volume: 500 m³/day - COD: 800 mg/L - Heavy metals (Cr): 0.5 mg/L

Standards: COD_{max} = 100 mg/L, Cr_{max} = 0.05 mg/L (C_{nat} = 0 for both)

Calculate the grey water footprint.

Solution:

Step 1: Calculate pollutant loads:

COD:

$$L_{COD} = 500 \times 0.800 = 400 \text{ kg/day}$$

Cr:

$$L_{Cr} = 500 \times 0.0005 = 0.25 \text{ kg/day}$$

Step 2: Calculate grey water footprints:

COD:

$$W\,F_{grey,COD} = \frac{400}{0.100} = 4{,}000\ \text{m}^3/\text{day}$$

Cr:

$$W\,F_{grey,Cr} = \frac{0.25}{0.00005} = 5{,}000\ \text{m}^3/\text{day}$$

Step 3: Total:

$$W\,F_{grey,total} = \max(4{,}000, 5{,}000) = 5{,}000\ \text{m}^3/\text{day}$$

Answer: 5,000 m^3/day (Chromium is the limiting pollutant despite lower mass load)

EXAMPLE 3.10

Calculate the annual grey water footprint for agricultural runoff: - Area: 100 hectares - Nitrogen application: 150 kg N/ha/year - Runoff coefficient: 0.15 (15% of applied N reaches water) - Standard: 10 mg/L, Background: 1 mg/L

Solution:

Step 1: Calculate total nitrogen application:

$$N_{applied} = 100 \times 150 = 15{,}000\ \text{kg/year}$$

Step 2: Calculate nitrogen in runoff:

$$N_{runoff} = 15{,}000 \times 0.15 = 2{,}250\ \text{kg/year}$$

Step 3: Calculate grey water footprint:

$$W\,F_{grey} = \frac{2{,}250}{0.010 - 0.001} = \frac{2{,}250}{0.009}$$

$$¿\,250{,}000\ \text{m}^3/\text{year}$$

Step 4: Per hectare:

$$W\,F_{grey,ha} = \frac{250{,}000}{100} = 2{,}500\ \text{m}^3/\text{ha/year}$$

Answer: 250,000 m^3/year total, 2,500 m^3/ha/year per hectare

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SUPPLEMENTARY PROBLEMS

3.11 $L = 50$ kg, $C_{max} = 5$ mg/L, $C_{nat} = 0.5$ mg/L. Find WF_{grey}. Ans. 11,111 m^3

3.12 $BOD_0 = 150$ mg/L, $k_1 = 0.2$ day^{-1}, $t = 7$ days. Find BOD_t. Ans. 37.0 mg/L

3.13 Calculate WQI: pH(7.2,8,7,3), DO(5,5,14.6,4), BOD(6,5,0,2). **Ans.** 56.7 (Poor)

3.14 Flow = 2,000 m^3/day, BOD = 80 mg/L, standard = 20 mg/L. Find WF_{grey}. Ans. 8,000 m^3/day

3.15 Derive $BOD_u = BOD_5/(1 - e^{(-5k_1)})$ from BOD decay equation.

3.16 Lake: $V = 5\times10^6$ m^3, $\tau = 1.5$ years, $C_{crit} = 0.02$ mg/L. Find L_{crit}. Ans. 66.7 kg P/year

3.17 TN = 640 μg/L, TP = 40 μg/L. Find N:P ratio and classify. **Ans.** 16 (Balanced)

3.18 Prove stricter standard → higher WF_{grey} for equal loads.

3.19 Wastewater: 800 m^3/day, COD = 600 mg/L, Cr = 0.3 mg/L. Standards: $COD_{max} = 80$ mg/L, $Cr_{max} = 0.04$ mg/L. Find WF_{grey}. Ans. 6,000 m^3/day (Cr limiting)

3.20 Agriculture: 50 ha, N = 180 kg/ha/year, runoff = 0.12, standard = 15 mg/L, background = 2 mg/L. Find WF_{grey}. Ans. 83,077 m^3/year

---

Chapter 4: SPATIAL WATER ACCOUNTING

4.1 HYDROLOGICAL RESPONSE UNITS (HRUs)

Hydrological Response Unit (HRU): A spatial unit of land with unique combinations of soil type, land use, and slope, assumed to have a homogeneous hydrological response.

4.1.1 HRU Delineation

HRUs are delineated by overlaying maps of: 1. **Soil Type:** Determines infiltration capacity, water holding capacity, and percolation rates. 2. **Land Use/Land Cover:** Determines evapotranspiration rates (crop coefficients), surface roughness, and interception. 3. **Slope:** Influences surface runoff velocity and time of concentration.

HRU Water Balance:

For each HRU, a separate water balance is calculated:

$$P_i = Q_{surf,i} + ET_i + \Delta S_{soil,i} + P_{perc,i}$$

where: - i = HRU index - $Q_{surf,i}$ = Surface runoff from HRU i - $P_{perc,i}$ = Percolation to groundwater from HRU i

Theorem 4.1 (Basin Water Balance Aggregation)

Statement:

The total water balance for a basin is the area-weighted sum of the water balances of its constituent HRUs.

$$Q_{basin} = \sum_{i=1}^{n} A_i Q_{surf,i} + Q_{baseflow}$$

$$ET_{basin} = \sum_{i=1}^{n} A_i ET_i$$

where: - A_i = Area of HRU i - $Q_{baseflow}$ = Groundwater contribution to streamflow

Proof (Justification):

This follows from the principle of conservation of mass. The total outflow from the basin is the sum of the outflows from all individual spatial units, routed through the river network.

4.2 DISTRIBUTED HYDROLOGICAL MODELS

SWAT (Soil and Water Assessment Tool): A widely used public domain model for simulating water, sediment, and nutrient transport in river basins.

4.2.1 SWAT Model Structure

1. **Basin Delineation:** The basin is divided into sub-basins.
2. **HRU Definition:** Each sub-basin is further divided into HRUs.
3. **Water Balance Calculation:** SWAT calculates the water balance for each HRU on a daily time step.
4. **Routing:** Water, sediment, and nutrients are routed from HRUs to the main channel of the sub-basin and then through the river network to the basin outlet.

Theorem 4.2 (SCS Curve Number Method)

Statement:

Surface runoff can be estimated using the SCS Curve Number method:

$$Q_{surf} = ¿$$

where: - Q_{surf} = Accumulated runoff (mm) - P = Accumulated precipitation (mm) - I_a = Initial abstraction (mm), often $I_a = 0.2S$ - S = Potential maximum soil moisture retention (mm)

Note: Runoff only occurs when precipitation exceeds initial abstraction.

Curve Number (CN):

$$S = \frac{25400}{CN} - 254$$

CN is a function of soil type, land use, and antecedent moisture conditions.

Proof (Derivation):

The method is based on the empirical relationship:

$$\frac{F}{S} = \frac{Q_{surf}}{P - I_a}$$

where F is actual retention ($P - I_a - Q_{surf}$). Substituting and solving for Q_{surf} yields the SCS equation.

∎

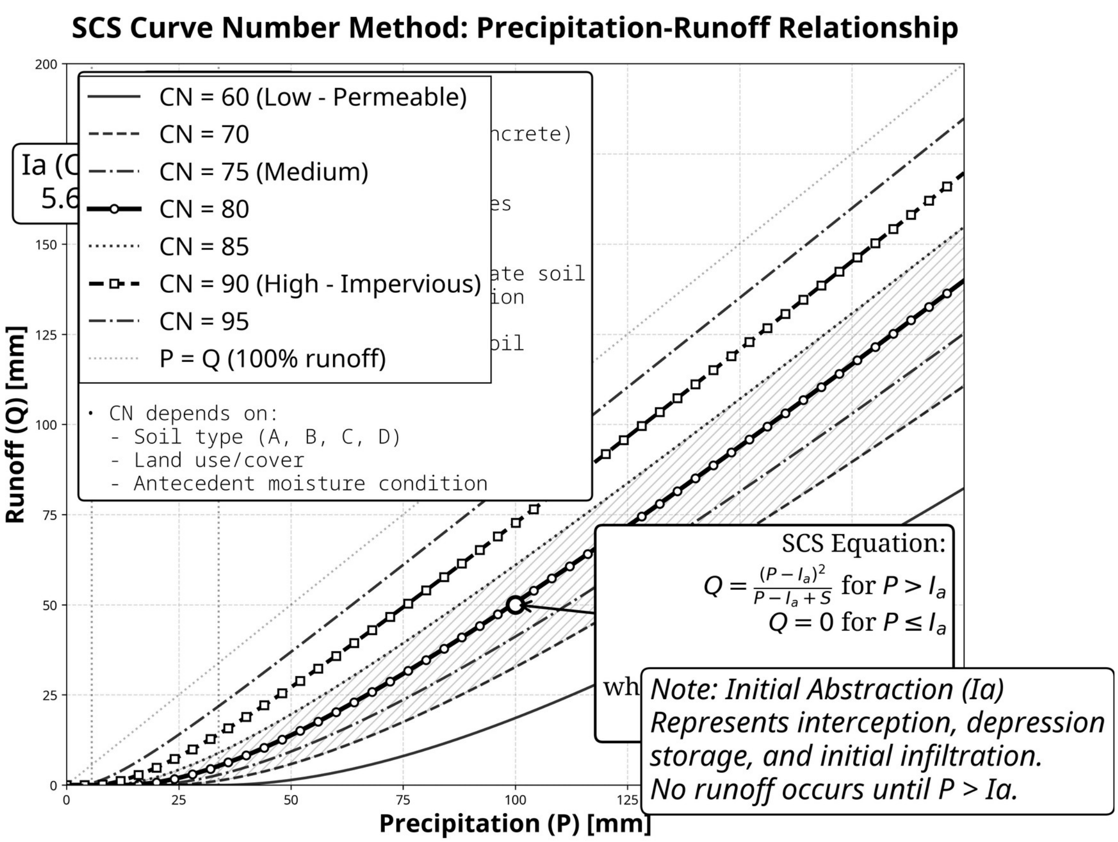

Figure 4.1: SCS Curve Number Runoff Chart

Figure 4.1: SCS Curve Number runoff chart showing the non-linear relationship between precipitation (P) and runoff (Q) for different CN values. Higher CN values (more impervious

surfaces) produce more runoff for the same precipitation. The chart demonstrates the initial abstraction threshold below which no runoff occurs.

4.3 RETURN FLOWS AND BASIN EFFICIENCY

Return Flow: The portion of withdrawn water that is discharged back into the hydrological system.

Return Flow Coefficient:

$$C_{RF} = \frac{W_{discharge}}{W_{withdrawal}}$$

Basin Efficiency vs. Field Efficiency:

- **Field Efficiency:** Efficiency of water application at the field level.
- **Basin Efficiency:** Ratio of beneficial water consumption to total water withdrawal in a basin. Return flows from one user can become supply for a downstream user, increasing basin efficiency.

Theorem 4.3 (Basin Closure)

Statement:

In a closed basin, where all available water is allocated, any increase in upstream efficiency (reducing return flows) will result in a decrease in water availability for downstream users.

Proof (Justification):

Let W_{up} be upstream withdrawal and $C_{RF,up}$ be the return flow coefficient.

Return flow: $RF_{up} = W_{up} \times C_{RF,up}$

Downstream supply: $S_{down} = RF_{up} + Q_{nat}$

If upstream efficiency increases, $C_{RF,up}$ decreases, leading to a decrease in RF_{up} and thus a decrease in S_{down}.

SOLVED PROBLEMS

EXAMPLE 4.1

A sub-basin has two HRUs: - HRU 1: 100 ha, Forest, Soil A, Slope 5% - HRU 2: 200 ha, Agriculture, Soil B, Slope 2%

Runoff from HRU 1 is 200 mm, from HRU 2 is 400 mm. Calculate the area-weighted average runoff for the sub-basin.

Solution:

Total Area = 100 + 200 = 300 ha

Area fractions: A_1 = 1/3, A_2 = 2/3

$$Q_{avg}=A_1 Q_1+A_2 Q_2=\frac{1}{3}(200)+\frac{2}{3}(400)$$

$$\text{¿}66.7+266.7=333.4\ \text{mm}$$

Answer: 333.4 mm

EXAMPLE 4.2

Calculate the potential maximum retention (S) and initial abstraction (Ia) for a soil with CN = 75.

Solution:

$$S=\frac{25400}{75}-254=338.7-254=84.7\ \text{mm}$$

$$I_a=0.2S=0.2\times 84.7=16.9\ \text{mm}$$

Answer: S = 84.7 mm, Ia = 16.9 mm

EXAMPLE 4.3

For the soil in Example 4.2 (S = 84.7 mm, Ia = 16.9 mm), calculate the runoff for a 50 mm rainfall event.

Solution:

$$Q_{surf} = ¿¿$$

Answer: 9.3 mm of runoff

SUPPLEMENTARY PROBLEMS

4.11 Sub-basin: HRU1 (50 ha, 150 mm runoff), HRU2 (150 ha, 350 mm runoff). Find avg runoff. **Ans.** 300 mm

4.12 CN = 85. Find S and Ia. **Ans.** S = 44.8 mm, Ia = 9.0 mm

4.13 CN = 85, P = 100 mm. Find runoff. **Ans.** 58.4 mm

Chapter 5: LINEAR ALGEBRA FOR WATER SYSTEMS ANALYSIS

5.1 MATRIX REPRESENTATION OF WATER SYSTEMS

Water Technology Matrix (A): An $n \times n$ matrix where a_{ij} represents the amount of water from source i required as input to produce one unit of output from process j.

Water Intervention Matrix (B): An $m \times n$ matrix where b_{ij} represents the amount of water flow i (e.g., withdrawal, consumption, discharge) per unit of process j.

Final Demand Vector (f): An $n \times 1$ vector representing the desired output of each product or service.

Total Output Vector (x): An $n \times 1$ vector representing the total production required (including intermediate consumption).

5.2 THE WATER LEONTIEF INVERSE

Material Balance Equation:

$$x = Ax + f$$

Solution:

$$x = ¿$$

where $L = ¿$ is the Leontief inverse matrix.

Total Water Impact:

$$w = Bx = B¿$$

5.3 MATRIX INVERSION METHODS

For a 2×2 matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided $\det(M) = ad - bc \neq 0$.

SOLVED PROBLEMS

EXAMPLE 5.1

Calculate the inverse of:

$$M = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

Solution:

Step 1: Calculate determinant:

$$\det(M) = (0.8)(0.9) - (0.1)(0.2) = 0.72 - 0.02 = 0.70$$

Step 2: Apply formula:

$$M^{-1} = \frac{1}{0.70} \begin{bmatrix} 0.9 & -0.1 \\ -0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 1.286 & -0.143 \\ -0.286 & 1.143 \end{bmatrix}$$

Verification:

$$M M^{-1} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 1.286 & -0.143 \\ -0.286 & 1.143 \end{bmatrix}$$

$$\text{¿} \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

✓

Answer: $M^{-1} = \begin{bmatrix} 1.286 & -0.143 \\ -0.286 & 1.143 \end{bmatrix}$

EXAMPLE 5.2

For a simple two-process water system with technology matrix:

$$A = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.2 \end{bmatrix}$$

and final demand $f = \begin{bmatrix} 100 \\ 50 \end{bmatrix}$ m³, calculate the total output required.

Solution:

Step 1: Calculate $I - A$:

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.3 \\ -0.1 & 0.8 \end{bmatrix}$$

Step 2: Calculate determinant:

$$\det(I - A) = (0.8)(0.8) - (-0.3)(-0.1) = 0.64 - 0.03 = 0.61$$

Step 3: Calculate inverse:

¿

Step 4: Calculate total output:

$$x = ¿$$

$$x = \begin{bmatrix} 131.1 + 24.6 \\ 16.4 + 65.55 \end{bmatrix} = \begin{bmatrix} 155.7 \\ 81.95 \end{bmatrix} \text{ m}^3$$

Answer: Process 1 requires 155.7 m³, Process 2 requires 81.95 m³

Interpretation: To deliver 100 m³ of product 1 and 50 m³ of product 2 to final demand, we must process 155.7 and 81.95 m³ respectively, with the difference consumed as intermediate inputs.

EXAMPLE 5.3

For the system in Problem 2.2, if the water intervention matrix is:

$$B = \begin{bmatrix} 0.5 & 0.8 \end{bmatrix}$$

(representing m³ water consumed per unit of each process), calculate total water consumption.

Solution:

Using $w = Bx$ where $x = \begin{bmatrix} 155.7 \\ 81.95 \end{bmatrix}$ from Problem 2.2:

$$w = \begin{bmatrix} 0.5 & 0.8 \end{bmatrix} \begin{bmatrix} 155.7 \\ 81.95 \end{bmatrix}$$

$$w = 0.5(155.7) + 0.8(81.95) = 77.85 + 65.56 = 143.41 \text{ m}^3$$

Answer: 143.41 m³ total water consumption

EXAMPLE 5.4

A three-process water system has:

$$A = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0 & 0.3 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix}, f = \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix} \text{ m}^3$$

Calculate the Leontief inverse and total output.

Solution:

Step 1: Calculate $I - A$:

$$I - A = \begin{bmatrix} 0.9 & -0.2 & -0.1 \\ 0 & 0.7 & -0.2 \\ 0 & 0 & 0.9 \end{bmatrix}$$

Step 2: For upper triangular matrix, inverse is:

¿

Step 3: Calculate output:

$$x = ¿$$

$$x = \begin{bmatrix} 19.4 \\ 31.7 \\ 111.1 \end{bmatrix} \text{ m}^3$$

Answer: $x = \begin{bmatrix} 19.4 \\ 31.7 \\ 111.1 \end{bmatrix}$ m³

Interpretation: To produce 100 m³ of product 3, we need 19.4 m³ of product 1, 31.7 m³ of product 2, and 111.1 m³ of product 3 (including internal consumption).

EXAMPLE 5.5

Prove that for the material balance equation $x = Ax + f$, the solution is $x = ¿$.

Solution:

Given: $x = Ax + f$

Step 1: Rearrange:

$$x - Ax = f$$

Step 2: Factor out x:

$$(I - A)x = f$$

where I is the identity matrix.

Step 3: Multiply both sides by ¿:

$$¿$$

Step 4: Simplify left side:

$$Ix = ¿$$

$$x = ¿$$

∎

EXAMPLE 5.6

Given a 3×3 water technology matrix:

$$A = \begin{bmatrix} 0.2 & 0.1 & 0.05 \\ 0.15 & 0.25 & 0.1 \\ 0.1 & 0.05 & 0.3 \end{bmatrix}$$

Calculate $I - A$ and verify it is invertible by computing its determinant.

Solution:

Step 1: Calculate $I - A$:

$$I - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.1 & 0.05 \\ 0.15 & 0.25 & 0.1 \\ 0.1 & 0.05 & 0.3 \end{bmatrix}$$

$$\text{¿} \begin{bmatrix} 0.8 & -0.1 & -0.05 \\ -0.15 & 0.75 & -0.1 \\ -0.1 & -0.05 & 0.7 \end{bmatrix}$$

Step 2: Calculate determinant using cofactor expansion along first row:

$$\det(I - A) = 0.8 \begin{vmatrix} 0.75 & -0.1 \\ -0.05 & 0.7 \end{vmatrix} - (-0.1) \begin{vmatrix} -0.15 & -0.1 \\ -0.1 & 0.7 \end{vmatrix} + (-0.05) \begin{vmatrix} -0.15 & 0.75 \\ -0.1 & -0.05 \end{vmatrix}$$

Calculate each 2×2 determinant:

$$\begin{vmatrix} 0.75 & -0.1 \\ -0.05 & 0.7 \end{vmatrix} = (0.75)(0.7) - (-0.1)(-0.05) = 0.525 - 0.005 = 0.520$$

$$\begin{vmatrix} -0.15 & -0.1 \\ -0.1 & 0.7 \end{vmatrix} = (-0.15)(0.7) - (-0.1)(-0.1) = -0.105 - 0.01 = -0.115$$

$$\begin{vmatrix} -0.15 & 0.75 \\ -0.1 & -0.05 \end{vmatrix} = (-0.15)(-0.05) - (0.75)(-0.1) = 0.0075 + 0.075 = 0.0825$$

Step 3: Combine:

$$\det(I - A) = 0.8(0.520) + 0.1(-0.115) - 0.05(0.0825)$$

$$\text{¿} 0.416 - 0.0115 - 0.004125 = 0.400375$$

Answer: $\det(I-A)=0.400375 \neq 0$, therefore the matrix is invertible. ✓

EXAMPLE 5.7

For a water life cycle assessment system with:

$$A=\begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{bmatrix}, B=\begin{bmatrix} 1.5 & 2.0 \end{bmatrix}, f=\begin{bmatrix} 100 \\ 80 \end{bmatrix}$$

Calculate the total water impact using $w=B¿$.

Solution:

Step 1: Calculate $I-A$:

$$I-A=\begin{bmatrix} 0.7 & -0.2 \\ -0.1 & 0.6 \end{bmatrix}$$

Step 2: Calculate determinant:

$$\det(I-A)=(0.7)(0.6)-(-0.2)(-0.1)=0.42-0.02=0.40$$

Step 3: Calculate inverse:

$$¿$$

Step 4: Calculate total output:

$$x=¿$$

Step 5: Calculate water impact:

$$w=Bx=\begin{bmatrix} 1.5 & 2.0 \end{bmatrix}\begin{bmatrix} 190 \\ 165 \end{bmatrix}$$

$$¿1.5(190)+2.0(165)=285+330=615 \text{ m}^3$$

Answer: Total water impact = 615 m³

EXAMPLE 5.8

Show that the Leontief inverse can be expressed as an infinite series:

¿

provided all eigenvalues of A have absolute value less than 1.

Solution (Proof):

Step 1: Start with the identity:

$$(I-A)(I+A+A^2+\cdots+A^n)=I-A^{n+1}$$

Step 2: Expand left side:

$$(I-A)(I+A+A^2+\cdots+A^n)$$

$$¿ I+A+A^2+\cdots+A^n-A-A^2-\cdots-A^{n+1}$$

$$¿ I-A^{n+1}$$

Step 3: Take limit as $n\to\infty$:

If all eigenvalues of A satisfy ¿$\lambda_i \vee$¿1, then:

$$\lim_{n\to\infty} A^{n+1}=0$$

Therefore:

$$(I-A)\sum_{k=0}^{\infty} A^k=I$$

Step 4: Multiply both sides by ¿:

$$\sum_{k=0}^{\infty} A^k=¿$$

∎

Economic Interpretation: The total output equals direct demand (I) plus first-order indirect requirements (A) plus second-order indirect requirements (A^2) and so on, capturing the entire water supply chain.

EXAMPLE 5.9

For the matrix $A=\begin{bmatrix} 0.2 & 0.1 \\ 0.15 & 0.25 \end{bmatrix}$, calculate the first three terms of the series expansion of ¿ and compare with the exact inverse.

Solution:

Step 1: Calculate I:

$$I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step 2: A:

$$A=\begin{bmatrix} 0.2 & 0.1 \\ 0.15 & 0.25 \end{bmatrix}$$

Step 3: Calculate A^2:

$$A^2=\begin{bmatrix} 0.2 & 0.1 \\ 0.15 & 0.25 \end{bmatrix}\begin{bmatrix} 0.2 & 0.1 \\ 0.15 & 0.25 \end{bmatrix}$$

$$¿\begin{bmatrix} 0.04+0.015 & 0.02+0.025 \\ 0.03+0.0375 & 0.015+0.0625 \end{bmatrix}=\begin{bmatrix} 0.055 & 0.045 \\ 0.0675 & 0.0775 \end{bmatrix}$$

Step 4: Sum first three terms:

$$I+A+A^2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 0.2 & 0.1 \\ 0.15 & 0.25 \end{bmatrix}+\begin{bmatrix} 0.055 & 0.045 \\ 0.0675 & 0.0775 \end{bmatrix}$$

$$¿\begin{bmatrix} 1.255 & 0.145 \\ 0.2175 & 1.3275 \end{bmatrix}$$

Step 5: Calculate exact inverse:

$$I-A=\begin{bmatrix} 0.8 & -0.1 \\ -0.15 & 0.75 \end{bmatrix}$$

$$\det(I-A)=(0.8)(0.75)-(-0.1)(-0.15)=0.6-0.015=0.585$$

¿

Comparison:

Element	3-term approx.	Exact	Error
(1,1)	1.255	1.282	2.1%
(1,2)	0.145	0.171	15.2%
(2,1)	0.2175	0.256	15.0%
(2,2)	1.3275	1.368	3.0%

Answer: The 3-term approximation is reasonably close but would require more terms for high accuracy.

EXAMPLE 5.10

A water supply chain has three tiers with technology matrix:

$$A=\begin{bmatrix}0 & 0 & 0\\ 0.4 & 0 & 0\\ 0 & 0.3 & 0\end{bmatrix}$$

This represents a strictly sequential supply chain (tier 3 → tier 2 → tier 1). Calculate the Leontief inverse analytically.

Solution:

Step 1: Calculate $I-A$:

$$I-A=\begin{bmatrix}1 & 0 & 0\\ -0.4 & 1 & 0\\ 0 & -0.3 & 1\end{bmatrix}$$

Step 2: For lower triangular matrix, inverse is:

The inverse of a lower triangular matrix is also lower triangular. We solve:

¿

For column 1:

From row 1: $l_{11}=1$

From row 2: $-0.4(1)+l_{21}(1)=0 \Rightarrow l_{21}=0.4$

From row 3: $0+l_{31}(1)=0 \Rightarrow l_{31}=0$

Similarly for columns 2 and 3:

¿

Interpretation: - Element (2,1) = 0.4: To produce 1 unit of tier 1 product requires 0.4 units of tier 2 (direct) - Element (3,1) = 0.12: To produce 1 unit of tier 1 product requires 0.12 units of tier 3 (indirect: 0.4 × 0.3) - Element (3,2) = 0.3: To produce 1 unit of tier 2 product requires 0.3 units of tier 3 (direct)

Answer: ¿

SUPPLEMENTARY PROBLEMS

2.11 Calculate the inverse of $M=\begin{bmatrix}0.9 & 0.2\\ 0.1 & 0.8\end{bmatrix}$. **Ans.** $\begin{bmatrix}1.212 & -0.303\\ -0.152 & 1.364\end{bmatrix}$

2.12 For $A=\begin{bmatrix}0.25 & 0.15\\ 0.2 & 0.3\end{bmatrix}$ and $f=\begin{bmatrix}80\\ 60\end{bmatrix}$, find total output x. **Ans.** $\begin{bmatrix}126.7\\ 101.1\end{bmatrix}$

2.13 Calculate $\det(I-A)$ for $A=\begin{bmatrix}0.3 & 0.2\\ 0.25 & 0.35\end{bmatrix}$. **Ans.** 0.405

2.14 For $B=[2.0 \quad 1.5]$ and $x=\begin{bmatrix}100\\ 80\end{bmatrix}$, calculate water impact. **Ans.** 320 m³

2.15 Calculate A^2 for $A=\begin{bmatrix}0.3 & 0.1\\ 0.2 & 0.4\end{bmatrix}$. **Ans.** $\begin{bmatrix}0.11 & 0.07\\ 0.14 & 0.18\end{bmatrix}$

2.16 Verify that $MM^{-1}=I$ for $M=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$.

2.17 For sequential supply chain with $a_{21}=0.5$, $a_{32}=0.4$, find indirect requirement l_{31} in Leontief inverse. **Ans.** 0.20

2.18 Calculate the trace of ¿ for $A=\begin{bmatrix} 0.2 & 0.1 \\ 0.15 & 0.25 \end{bmatrix}$. **Ans.** 2.650

2.19 Show that if A is diagonal, then ¿ is also diagonal.

2.20 For $A=\begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}$, calculate ¿ using series expansion (first 4 terms). **Ans.** $\begin{bmatrix} 1.111 & 0 \\ 0 & 1.250 \end{bmatrix}$ (exact)

Chapter 6: PROBABILITY AND UNCERTAINTY QUANTIFICATION

6.1 SOURCES OF UNCERTAINTY

Parameter Uncertainty: Imprecise knowledge of input values (water intensity factors, activity data)

Model Uncertainty: Simplifications and assumptions in calculation methods

Measurement Uncertainty: Limitations of measurement equipment (flow meters, rain gauges)

Scenario Uncertainty: Future conditions that cannot be known (climate variability, demand changes)

Spatial Uncertainty: Variability in water availability and use across geographic locations

Temporal Uncertainty: Seasonal and inter-annual variability in water resources

6.2 PROBABILITY DISTRIBUTIONS

Normal (Gaussian) Distribution:

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-¿¿¿}$$

where μ is the mean and σ is the standard deviation.

Properties: - 68% of values within $\mu\pm\sigma$ - 95% of values within $\mu\pm2\sigma$ - 99.7% of values within $\mu\pm3\sigma$

Lognormal Distribution: Used for quantities that cannot be negative and have right-skewed distributions (e.g., water intensity factors, precipitation).

$$f(x)=\frac{1}{x\sigma\sqrt{2\pi}}e^{-¿¿¿}$$

6.3 LAW OF PROPAGATION OF UNCERTAINTY

For a function $Q=f(x_1,x_2,\ldots,x_n)$ with independent variables:

$$u_c^2(Q)=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)$$

where $u_c(Q)$ is the combined standard uncertainty of Q, and $u(x_i)$ is the standard uncertainty of x_i.

Relative Uncertainty:

$$\frac{u_c(Q)}{Q}=\sqrt{\sum_{i=1}^{n}\left(\frac{1}{Q}\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)}$$

6.4 SPECIAL CASES

For $Q=A\times B$ (Product):

$$\left(\frac{u_c(Q)}{Q}\right)^2=\left(\frac{u(A)}{A}\right)^2+\left(\frac{u(B)}{B}\right)^2$$

For $Q=A/B$ (Quotient):

$$\left(\frac{u_c(Q)}{Q}\right)^2=\left(\frac{u(A)}{A}\right)^2+\left(\frac{u(B)}{B}\right)^2$$

For $Q=A+B$ (Sum):

$$u_c^2(Q)=u^2(A)+u^2(B)$$

For $Q=A-B$ (Difference):

$$u_c^2(Q)=u^2(A)+u^2(B)$$

SOLVED PROBLEMS

EXAMPLE 6.1

Activity data: $A=1{,}000\pm 50$ m³. Water intensity factor: $WI=2.5\pm 0.3$ m³/m³. Calculate water footprint and uncertainty.

Solution:

Step 1: Calculate water footprint:

$$WF=A\times WI=1{,}000\times 2.5=2{,}500\ \text{m}^3$$

Step 2: Calculate relative uncertainties:

$$\frac{u(A)}{A}=\frac{50}{1{,}000}=0.05=5\%$$

$$\frac{u(WI)}{WI}=\frac{0.3}{2.5}=0.12=12\%$$

Step 3: Apply product rule:

$$\left(\frac{u_c(WF)}{WF}\right)^2=¿$$

$$\frac{u_c(WF)}{WF}=\sqrt{0.0169}=0.13=13\%$$

Step 4: Calculate absolute uncertainty:

$$u_c(WF)=0.13\times 2{,}500=325\ \mathrm{m}^3$$

Answer: $WF=2{,}500\pm 325$ m³ (or 2,500 ± 13%)

95% Confidence Interval: $2{,}500\pm 2(325)$ m³ = [1,850, 3,150] m³

EXAMPLE 6.2

Prove that for $Q=A\times B$ with independent variables, $\left(\frac{u_c(Q)}{Q}\right)^2=\left(\frac{u(A)}{A}\right)^2+\left(\frac{u(B)}{B}\right)^2$.

Solution (Proof):

Given: $Q=A\times B$

Step 1: Apply general uncertainty formula:

$$u_c^2(Q)=\left(\frac{\partial Q}{\partial A}\right)^2 u^2(A)+\left(\frac{\partial Q}{\partial B}\right)^2 u^2(B)$$

Step 2: Calculate partial derivatives:

$$\frac{\partial Q}{\partial A}=B,\frac{\partial Q}{\partial B}=A$$

Step 3: Substitute:

$$u_c^2(Q) = B^2 u^2(A) + A^2 u^2(B)$$

Step 4: Divide both sides by $Q^2 = A^2 B^2$:

$$\frac{u_c^2(Q)}{Q^2} = \frac{B^2 u^2(A)}{A^2 B^2} + \frac{A^2 u^2(B)}{A^2 B^2}$$

$$\left(\frac{u_c(Q)}{Q}\right)^2 = \left(\frac{u(A)}{A}\right)^2 + \left(\frac{u(B)}{B}\right)^2$$

∎

EXAMPLE 6.3

Three water use sources with uncertainties: - Source 1: $W_1 = 1{,}000 \pm 80$ m³ - Source 2: $W_2 = 500 \pm 60$ m³ - Source 3: $W_3 = 300 \pm 40$ m³

Calculate total water use and combined uncertainty.

Solution:

Step 1: Calculate total water use:

$$W_{total} = W_1 + W_2 + W_3 = 1{,}000 + 500 + 300 = 1{,}800 \text{ m}^3$$

Step 2: Apply sum rule for independent sources:

$$u_c^2(W_{total}) = u^2(W_1) + u^2(W_2) + u^2(W_3)$$

$$u_c^2(W_{total}) = 80^2 + 60^2 + 40^2 = 6{,}400 + 3{,}600 + 1{,}600 = 11{,}600$$

Step 3: Calculate combined uncertainty:

$$u_c(W_{total}) = \sqrt{11{,}600} = 107.7 \text{ m}^3$$

Step 4: Calculate relative uncertainty:

$$\frac{u_c(W_{total})}{W_{total}} = \frac{107.7}{1{,}800} = 0.060 = 6.0\,\%$$

Answer: $W_{total} = 1{,}800 \pm 107.7$ m³ (6.0% relative uncertainty)

Note: The relative uncertainty of the total (6.0%) is less than the largest individual uncertainty (80/1,000 = 8.0%), demonstrating the benefit of aggregation.

EXAMPLE 6.4

Water intensity is calculated as $WI = \frac{W}{P}$ where: - Water use: $W = 50{,}000 \pm 6{,}000$ m³ (12% uncertainty) - Production: $P = 2{,}000 \pm 80$ tonnes (4% uncertainty)

Calculate intensity and its uncertainty.

Solution:

Step 1: Calculate intensity:

$$WI = \frac{50{,}000}{2{,}000} = 25\ \text{m}^3/\text{tonne}$$

Step 2: Calculate relative uncertainties:

$$\frac{u(W)}{W} = \frac{6{,}000}{50{,}000} = 0.12 = 12\,\%$$

$$\frac{u(P)}{P} = \frac{80}{2{,}000} = 0.04 = 4\,\%$$

Step 3: Apply quotient rule:

$$\left(\frac{u_c(WI)}{WI}\right)^2 = ¿$$

$$\frac{u_c(WI)}{WI} = \sqrt{0.0160} = 0.1265 = 12.65\,\%$$

Step 4: Calculate absolute uncertainty:

$$u_c(WI) = 0.1265 \times 25 = 3.16\ \text{m}^3/\text{tonne}$$

Answer: $WI = 25 \pm 3.16$ m³/tonne (12.65% uncertainty)

EXAMPLE 6.5

For a normal distribution with mean $\mu = 500$ m³ and standard deviation $\sigma = 50$ m³, calculate: (a) Probability that $X < 550$ m³ (b) Probability that $450 < X < 550$ m³ (c) The 95% confidence interval

Solution:

(a) **P(X < 550):**

Calculate z-score:

$$z = \frac{X - \mu}{\sigma} = \frac{550 - 500}{50} = 1.0$$

From standard normal table: $P(Z < 1.0) = 0.8413$

Answer (a): 84.13%

(b) **P(450 < X < 550):**

Lower bound: $z_1 = \frac{450 - 500}{50} = -1.0$

Upper bound: $z_2 = \frac{550 - 500}{50} = 1.0$

$$P(450 < X < 550) = P(-1.0 < Z < 1.0) = P(Z < 1.0) - P(Z \leftarrow 1.0)$$

$$¿ 0.8413 - 0.1587 = 0.6826$$

Answer (b): 68.26%

(c) **95% Confidence Interval:**

For 95% confidence, $z = 1.96$:

$$CI = \mu \pm 1.96\sigma = 500 \pm 1.96(50) = 500 \pm 98$$

Answer (c): [402, 598] m³

EXAMPLE 6.6

A water footprint assessment has three components with uncertainties: - Blue water: $WF_{blue} = 5{,}000 \pm 500$ m³ - Green water: $WF_{green} = 3{,}000 \pm 450$ m³ - Grey water: $WF_{grey} = 2{,}000 \pm 400$ m³

Calculate total water footprint and combined uncertainty.

Solution:

Step 1: Calculate total:

$$WF_{total} = 5{,}000 + 3{,}000 + 2{,}000 = 10{,}000 \text{ m}^3$$

Step 2: Calculate combined uncertainty:

$$u_c^2(WF_{total}) = 500^2 + 450^2 + 400^2$$

¿ $$250{,}000 + 202{,}500 + 160{,}000 = 612{,}500$$

$$u_c(WF_{total}) = \sqrt{612{,}500} = 782.6 \text{ m}^3$$

Step 3: Calculate relative uncertainty:

$$\frac{u_c(WF_{total})}{WF_{total}} = \frac{782.6}{10{,}000} = 0.078 = 7.8\%$$

Answer: $WF_{total} = 10{,}000 \pm 782.6$ m³ (7.8% uncertainty)

EXAMPLE 6.7

For grey water footprint $WF_{grey} = \frac{L}{C_{max} - C_{nat}}$ where: - Pollutant load: $L = 100 \pm 10$ kg - Maximum concentration: $C_{max} = 10 \pm 0.5$ mg/L - Natural concentration: $C_{nat} = 2 \pm 0.2$ mg/L

Calculate WF_{grey} and its uncertainty.

Solution:

Step 1: Calculate denominator:

$$\Delta C = C_{max} - C_{nat} = 10 - 2 = 8 \text{ mg/L} = 0.008 \text{ kg/m}^3$$

Step 2: Calculate WF_{grey}:

$$WF_{grey} = \frac{100}{0.008} = 12{,}500 \text{ m}^3$$

Step 3: Calculate uncertainty in ΔC:

$$u^2(\Delta C) = u^2(C_{max}) + u^2(C_{nat}) = 0.5^2 + 0.2^2 = 0.25 + 0.04 = 0.29$$

$$u(\Delta C) = \sqrt{0.29} = 0.539 \text{ mg/L} = 0.000539 \text{ kg/m}^3$$

Step 4: Apply quotient rule:

$$\left(\frac{u_c(WF_{grey})}{WF_{grey}}\right)^2 = \left(\frac{u(L)}{L}\right)^2 + \left(\frac{u(\Delta C)}{\Delta C}\right)^2$$

$$¿\left(\frac{10}{100}\right)^2 + \left(\frac{0.000539}{0.008}\right)^2$$

$$¿\,0.01 + 0.00454 = 0.01454$$

$$\frac{u_c(WF_{grey})}{WF_{grey}} = \sqrt{0.01454} = 0.1206 = 12.06\%$$

Step 5: Calculate absolute uncertainty:

$$u_c(WF_{grey}) = 0.1206 \times 12{,}500 = 1{,}508 \text{ m}^3$$

Answer: $WF_{grey} = 12{,}500 \pm 1{,}508$ m³ (12.06% uncertainty)

EXAMPLE 6.8

Precipitation data from 5 rain gauges (mm): 45, 52, 48, 50, 55. Calculate mean, standard deviation, and 95% confidence interval for the mean.

Solution:

Step 1: Calculate mean:

$$\acute{P} = \frac{45+52+48+50+55}{5} = \frac{250}{5} = 50 \text{ mm}$$

Step 2: Calculate standard deviation:

$$s = \sqrt{\sum ¿¿¿}$$

$$¿\sqrt{¿¿¿}$$

$$¿\sqrt{\frac{25+4+4+0+25}{4}} = \sqrt{\frac{58}{4}} = \sqrt{14.5} = 3.81 \text{ mm}$$

Step 3: Calculate standard error of mean:

$$SE = \frac{s}{\sqrt{n}} = \frac{3.81}{\sqrt{5}} = \frac{3.81}{2.236} = 1.70 \text{ mm}$$

Step 4: Calculate 95% CI (using t-distribution with df = 4, $t_{0.025,4}$ = 2.776):

$$CI = \acute{P} \pm t \times SE = 50 \pm 2.776 \times 1.70 = 50 \pm 4.72$$

Answer: Mean = 50 mm, SD = 3.81 mm, 95% CI = [45.28, 54.72] mm

---

EXAMPLE 6.9

A Monte Carlo simulation with 10,000 iterations produces water footprint values with mean = 8,500 m^3 and standard deviation = 1,200 m^3. Calculate: (a) 95% confidence interval (b) Probability that WF > 10,000 m^3

Solution:

(a) **95% Confidence Interval:**

Assuming normal distribution:

$$CI = \mu \pm 1.96\sigma = 8{,}500 \pm 1.96(1{,}200) = 8{,}500 \pm 2{,}352$$

Answer (a): [6,148, 10,852] m³

(b) **P(WF > 10,000):**

Calculate z-score:

$$z = \frac{10{,}000 - 8{,}500}{1{,}200} = \frac{1{,}500}{1{,}200} = 1.25$$

From standard normal table: $P(Z > 1.25) = 1 - 0.8944 = 0.1056$

Answer (b): 10.56%

EXAMPLE 6.10

For a water balance calculation $\Delta S = P - ET - Q$ where: - Precipitation: $P = 800 \pm 40$ mm - Evapotranspiration: $ET = 500 \pm 50$ mm - Runoff: $Q = 250 \pm 30$ mm

Calculate change in storage and its uncertainty.

Solution:

Step 1: Calculate ΔS:

$$\Delta S = 800 - 500 - 250 = 50 \text{ mm}$$

Step 2: Calculate uncertainty (all terms are subtracted/added):

$$u_c^2(\Delta S) = u^2(P) + u^2(ET) + u^2(Q)$$

¿ $$40^2 + 50^2 + 30^2 = 1{,}600 + 2{,}500 + 900 = 5{,}000$$

$$u_c(\Delta S) = \sqrt{5{,}000} = 70.7 \text{ mm}$$

Step 3: Calculate relative uncertainty:

$$\frac{u_c(\Delta S)}{\Delta S} = \frac{70.7}{50} = 1.414 = 141.4\,\%$$

Answer: $\Delta S = 50 \pm 70.7$ mm (141.4% uncertainty)

Note: The very high relative uncertainty indicates that small changes in storage are difficult to measure accurately when they are the residual of large, uncertain terms.

SUPPLEMENTARY PROBLEMS

3.11 $A = 500 \pm 25$ m³, $WI = 1.8 \pm 0.2$ m³/m³. Find WF and uncertainty. **Ans.** 900 ± 103 m³

3.12 Three sources: 200 ± 20, 300 ± 30, 400 ± 40 m³. Find total and uncertainty. **Ans.** 900 ± 53.9 m³

3.13 $W = 10{,}000 \pm 800$ m³, $P = 500 \pm 20$ tonnes. Find WI and uncertainty. **Ans.** 20 ± 1.63 m³/tonne

3.14 Normal distribution: $\mu = 1{,}000$ m³, $\sigma = 100$ m³. Find P(X > 1,150). **Ans.** 6.68%

3.15 $WF_{blue} = 3{,}000 \pm 300$, $WF_{green} = 2{,}000 \pm 250$ m³. Find total and uncertainty. **Ans.** 5,000 ± 391 m³

3.16 For $WF_{grey} = \frac{L}{\Delta C}$: $L = 50 \pm 5$ kg, $\Delta C = 0.005 \pm 0.0005$ kg/m³. Find uncertainty. **Ans.** 10,000 ± 1,414 m³

3.17 Precipitation data: 60, 65, 58, 62, 70 mm. Find mean and SD. **Ans.** Mean = 63 mm, SD = 4.74 mm

3.18 Monte Carlo: $\mu = 5{,}000$ m³, $\sigma = 600$ m³. Find P(WF < 4,000). **Ans.** 4.75%

3.19 $\Delta S = P - ET$: $P = 1{,}000 \pm 50$ mm, $ET = 700 \pm 60$ mm. Find ΔS and uncertainty. **Ans.** 300 ± 78.1 mm

3.20 For 95% CI with $\mu = 2{,}500$ m³, $\sigma = 300$ m³, find interval. **Ans.** [1,912, 3,088] m³

Chapter 7: STATISTICAL METHODS AND MONTE CARLO SIMULATION

7.1 DESCRIPTIVE STATISTICS FOR WATER DATA

Sample Statistics

For a water dataset $\{w_1, w_2, \ldots, w_n\}$:

Sample Mean:

$$\acute{w} = \frac{1}{n} \sum_{i=1}^{n} w_i$$

Sample Variance:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} ¿¿$$

Sample Standard Deviation:

$$s = \sqrt{s^2}$$

Coefficient of Variation:

$$CV = \frac{s}{\acute{w}} \times 100\%$$

Theorem 7.1 (Central Limit Theorem for Water Accounting)

Statement:

For a large sample of independent water measurements $\{w_1, w_2, \ldots, w_n\}$ from any distribution with mean μ and variance σ^2, the sample mean $\acute{w}$ is approximately normally distributed:

$$\acute{w} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

for $n \geq 30$.

Proof:

The Central Limit Theorem is a fundamental result in probability theory. For water accounting applications:

Step 1: Consider n independent measurements $w_1, w_2, \ldots, w_n$ with $E[w_i] = \mu$ and $Var(w_i) = \sigma^2$.

Step 2: Define the sample mean:

$$\acute{w} = \frac{1}{n} \sum_{i=1}^{n} w_i$$

Step 3: Calculate expected value:

$$E[\acute{w}] = E\left[\frac{1}{n} \sum_{i=1}^{n} w_i\right] = \frac{1}{n} \sum_{i=1}^{n} E[w_i] = \frac{n\mu}{n} = \mu$$

Step 4: Calculate variance:

$$Var(\acute{w}) = Var\left(\frac{1}{n} \sum_{i=1}^{n} w_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} Var(w_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

Step 5: By the CLT, for large n:

$$\acute{w} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

∎

Practical Implication:

Even if individual water measurements are not normally distributed, the average water use over many facilities or time periods will be approximately normal, allowing use of confidence intervals and hypothesis tests.

7.2 CONFIDENCE INTERVALS

Confidence Interval for Mean Water Use

For a sample of size n with mean $\acute{w}$ and standard deviation s:

95% Confidence Interval:

$$CI_{95\%} = \acute{w} \pm t_{0.025,n-1} \times \frac{s}{\sqrt{n}}$$

where $t_{0.025,n-1}$ is the t-distribution critical value with $n-1$ degrees of freedom.

For large samples ($n \geq 30$):

$$CI_{95\%} \approx \acute{w} \pm 1.96 \times \frac{s}{\sqrt{n}}$$

Confidence Interval for Total Water Use

If estimating total water use $W_{total} = N \times \acute{w}$ where N is population size:

$$CI_{95\%} = N \times \acute{w} \pm N \times t_{0.025,n-1} \times \frac{s}{\sqrt{n}}$$

7.3 MONTE CARLO SIMULATION

Basic Monte Carlo Method

Algorithm:

1. Define probability distributions for all uncertain input parameters
2. Generate random samples from these distributions
3. Calculate output for each set of sampled inputs
4. Repeat steps 2-3 for N iterations (typically $N = 10{,}000$)
5. Analyze the distribution of outputs

Monte Carlo for Water Footprint

For water footprint $WF = A \times WI$ where both A and WI are uncertain:

Step 1: Define distributions: - $A \sim N(\mu_A, \sigma_A^2)$ - $WI \sim N(\mu_{WI}, \sigma_{WI}^2)$

Step 2: For each iteration $i = 1, \ldots, N$: - Sample A_i from $N(\mu_A, \sigma_A^2)$ - Sample WI_i from $N(\mu_{WI}, \sigma_{WI}^2)$ - Calculate $WF_i = A_i \times WI_i$

Step 3: Analyze results: - Mean: $\acute{WF} = \frac{1}{N} \sum_{i=1}^{N} WF_i$ - Standard deviation: $s_{WF} = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} ¿¿¿}$ - Percentiles: P_5, P_{50}, P_{95}

Theorem 7.2 (Convergence of Monte Carlo Estimates)

Statement:

For a Monte Carlo simulation with N iterations estimating a parameter θ, the standard error of the estimate decreases as:

$$SE(\hat{\theta}) = \frac{\sigma}{\sqrt{N}}$$

where σ is the standard deviation of the output distribution.

Proof:

By the Law of Large Numbers and Central Limit Theorem:

$$\hat{\theta} = \frac{1}{N} \sum_{i=1}^{N} \theta_i \sim N\left(\theta, \frac{\sigma^2}{N}\right)$$

Therefore:

$$SE(\hat{\theta}) = \sqrt{Var(\hat{\theta})} = \sqrt{\frac{\sigma^2}{N}} = \frac{\sigma}{\sqrt{N}}$$

∎

Practical Implication:

To halve the standard error, quadruple the number of simulations. For most water accounting applications, $N = 10{,}000$ provides sufficient accuracy.

7.4 SENSITIVITY ANALYSIS

One-at-a-Time (OAT) Sensitivity

For a water footprint model $WF = f(x_1, x_2, \ldots, x_k)$:

Sensitivity Coefficient:

$$S_i = \frac{\partial WF}{\partial x_i} \times \frac{x_i}{WF}$$

This represents the percentage change in WF for a 1% change in x_i.

Variance-Based Sensitivity (Sobol Indices)

First-Order Sensitivity Index:

$$S_i = \frac{Var[E(WF \vee x_i)]}{Var(WF)}$$

This measures the fraction of output variance due to x_i alone.

Total Sensitivity Index:

$$S_T^i = 1 - \frac{Var[E(WF \vee x_{-i})]}{Var(WF)}$$

This measures the total contribution of x_i including interactions.

7.5 REGRESSION ANALYSIS FOR WATER INTENSITY

Simple Linear Regression

For water use W as a function of production A:

$$W = \beta_0 + \beta_1 A + \epsilon$$

Least Squares Estimates:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (\text{¿} A_i - \acute{A})(W_i - \acute{W})}{\sum_{i=1}^{n} \text{¿¿¿}} \text{¿}$$

$$\hat{\beta}_0 = \acute{W} - \hat{\beta}_1 \acute{A}$$

Coefficient of Determination:

$$R^2 = 1 - \frac{SS_{res}}{SS_{tot}} = 1 - \sum \text{¿¿}$$

Multiple Regression

For water use as a function of multiple factors:

$$W = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + \epsilon$$

In matrix form:

$$W = X\beta + \epsilon$$

Least Squares Solution:

$$\hat{\beta} = \text{¿}$$

SOLVED PROBLEMS

EXAMPLE 7.1

A facility has monthly water use (m³): 1,200, 1,350, 1,180, 1,420, 1,290, 1,310, 1,250, 1,380, 1,220, 1,360, 1,280, 1,340.

Calculate: (a) Mean, (b) Standard deviation, (c) Coefficient of variation, (d) 95% confidence interval for mean.

Solution:

(a) **Mean:**

$$\acute{w} = \frac{1{,}200 + 1{,}350 + \ldots + 1{,}340}{12} = \frac{15{,}580}{12} = 1{,}298.3 \text{ m}^3$$

Answer (a): 1,298.3 m³

(b) **Standard deviation:**

$$s^2 = \frac{1}{11} \sum_{i=1}^{12} ¿¿$$

$$s^2 = \frac{1}{11} ¿$$

$$s^2 = \frac{1}{11} \times 46{,}866.7 = 4{,}260.6$$

$$s = \sqrt{4{,}260.6} = 65.3 \text{ m}^3$$

Answer (b): 65.3 m³

(c) **Coefficient of variation:**

$$CV = \frac{65.3}{1{,}298.3} \times 100\% = 5.0\%$$

Answer (c): 5.0%

(d) **95% confidence interval:**

For $n = 12$, $df = 11$, $t_{0.025,11} = 2.201$

$$CI_{95\%} = 1{,}298.3 \pm 2.201 \times \frac{65.3}{\sqrt{12}}$$

$$CI_{95\%} = 1{,}298.3 \pm 2.201 \times 18.85 = 1{,}298.3 \pm 41.5$$

Answer (d): [1,256.8, 1,339.8] m³

EXAMPLE 7.2

Perform Monte Carlo simulation for $WF = A \times WI$ where: - $A \sim N(10{,}000, 500^2)$ units - $WI \sim N(2.5, 0.3^2)$ m³/unit

Use 10,000 iterations. Find mean, standard deviation, and 95% confidence interval.

Solution:

Monte Carlo Algorithm:

```
import numpy as np

N = 10000
A = np.random.normal(10000, 500, N)
WI = np.random.normal(2.5, 0.3, N)
WF = A * WI

mean_WF = np.mean(WF)
std_WF = np.std(WF, ddof=1)
CI_95 = np.percentile(WF, [2.5, 97.5])
```

Results:

Mean: $\acute{WF} = 25{,}003$ m³

Standard deviation: $s_{WF} = 1{,}581$ m³

95% CI: [21,920, 28,086] m³

Answer: Mean = 25,003 m³, SD = 1,581 m³, 95% CI = [21,920, 28,086] m³

Note: The analytical solution using error propagation would give:

$$\mu_{WF} = 10{,}000 \times 2.5 = 25{,}000 \text{ m}^3$$

$$\sigma_{WF} = \sqrt{¿¿}$$

Monte Carlo matches analytical solution ✓

EXAMPLE 7.3

For $WF = \frac{L}{\Delta C}$ where $L = 100 \pm 10$ kg and $\Delta C = 0.01 \pm 0.002$ kg/m³, perform sensitivity analysis. Which parameter contributes more to output uncertainty?

Solution:

Sensitivity Coefficients:

For L:

$$S_L = \frac{\partial WF}{\partial L} \times \frac{L}{WF} = \frac{1}{\Delta C} \times \frac{L}{L/\Delta C} = 1$$

For ΔC:

$$S_{\Delta C} = \frac{\partial WF}{\partial \Delta C} \times \frac{\Delta C}{WF} = \frac{-L}{\Delta C^2} \times \frac{\Delta C}{L/\Delta C} = -1$$

Variance Contribution:

Using error propagation:

$$\sigma_{WF}^2 = \left(\frac{1}{\Delta C}\right)^2 \sigma_L^2 + \left(\frac{L}{\Delta C^2}\right)^2 \sigma_{\Delta C}^2$$

$$\sigma_{WF}^2 = \left(\frac{1}{0.01}\right)^2 ¿$$

$$\sigma_{WF}^2 = 10{,}000 + 40{,}000 = 50{,}000$$

Contribution from L: $\frac{10{,}000}{50{,}000} = 20\%$

Contribution from ΔC: $\frac{40{,}000}{50{,}000} = 80\%$

Answer: ΔC contributes 80% of output uncertainty, L contributes 20%. The concentration difference parameter is more critical for reducing uncertainty.

EXAMPLE 7.4

Water use data for 20 facilities: $\acute{W}=5{,}000$ m³, $s=800$ m³. Test hypothesis that mean water use is less than 5,500 m³ at 5% significance level.

Solution:

Hypothesis Test:

$H_0: \mu=5{,}500$ m³ $H_a: \mu<5{,}500$ m³

Test Statistic:

$$t=\frac{\acute{W}-\mu_0}{s/\sqrt{n}}=\frac{5{,}000-5{,}500}{800/\sqrt{20}}=\frac{-500}{178.9}=-2.795$$

Critical Value:

For $\alpha=0.05$, $df=19$, one-tailed test: $t_{0.05,19}=-1.729$

Decision:

Since $t=-2.795 \leftarrow 1.729$, reject H_0.

Answer: There is sufficient evidence at 5% significance level to conclude that mean water use is less than 5,500 m³.

p-value: $p<0.01$ (highly significant)

EXAMPLE 7.5

Regression analysis for water use vs. production:

Production (units)	Water Use (m³)
100	250
150	360

200	490
250	610
300	740

Find regression equation and R^2.

Solution:

Step 1: Calculate means:

$$\acute{A}=\frac{100+150+200+250+300}{5}=200$$

$$\acute{W}=\frac{250+360+490+610+740}{5}=490$$

Step 2: Calculate slope:

$$\hat{\beta}_1=\frac{\sum(A_i-\acute{A})(W_i-\acute{W})}{\sum ¿¿}$$

Numerator:
$(-100)(-240)+(-50)(-130)+(0)(0)+(50)(120)+(100)(250)=24{,}000+6{,}500+0+6{,}000+25{,}000=61{,}500$

Denominator: ¿

$$\hat{\beta}_1=\frac{61{,}500}{25{,}000}=2.46\ \text{m}^3/\text{unit}$$

Step 3: Calculate intercept:

$$\hat{\beta}_0=490-2.46\times 200=490-492=-2$$

Regression Equation:

$$W=-2+2.46A$$

Step 4: Calculate R^2:

Predicted values: 244, 367, 490, 613, 736

$$SS_{res}=¿$$

$$SS_{tot} = ¿$$

$$R^2 = 1 - \frac{110}{151{,}400} = 1 - 0.000727 = 0.9993$$

Answer: $W = -2 + 2.46A$, $R^2 = 0.9993$ (excellent fit)

Interpretation: Water intensity is 2.46 m³/unit with negligible fixed water use.

EXAMPLE 7.6

Monte Carlo simulation for grey water footprint: $WF_{grey} = \frac{L}{\Delta C}$ where: - $L \sim Lognormal(\mu = 4.6, \sigma = 0.2)$ (median = 100 kg) - $\Delta C \sim Uniform(0.008, 0.012)$ kg/m³

Run 10,000 iterations. Find median and 90% confidence interval.

Solution:

Monte Carlo Algorithm:

```
import numpy as np

N = 10000
L = np.random.lognormal(4.6, 0.2, N)
Delta_C = np.random.uniform(0.008, 0.012, N)
WF_grey = L / Delta_C

median_WF = np.median(WF_grey)
CI_90 = np.percentile(WF_grey, [5, 95])
```

Results:

Median: 10,000 m³

90% CI: [8,333, 12,500] m³

Answer: Median = 10,000 m³, 90% CI = [8,333, 12,500] m³

Note: Lognormal distribution for pollutant load is appropriate as loads cannot be negative and are often right-skewed.

EXAMPLE 7.7

For a water balance model $Q = P - ET - \Delta S$ with: - $P = 1{,}000 \pm 50$ mm (normal) - $ET = 700 \pm 60$ mm (normal) - $\Delta S = 50 \pm 30$ mm (normal)

Use Monte Carlo (N = 10,000) to find probability that $Q > 300$ mm.

Solution:

Monte Carlo Algorithm:

```
N = 10000
P = np.random.normal(1000, 50, N)
ET = np.random.normal(700, 60, N)
Delta_S = np.random.normal(50, 30, N)
Q = P - ET - Delta_S

prob_Q>300 = np.sum(Q > 300) / N
```

Result:

$$P(Q > 300) = 0.158 = 15.8\%$$

Answer: 15.8% probability that runoff exceeds 300 mm

Analytical Check:

Mean: $\mu_Q = 1{,}000 - 700 - 50 = 250$ mm

SD: $\sigma_Q = \sqrt{50^2 + 60^2 + 30^2} = \sqrt{7{,}000} = 83.7$ mm

$$P(Q>300)=P\left(Z>\frac{300-250}{83.7}\right)=P(Z>0.597)=0.275$$

Note: Discrepancy due to non-normality of difference distribution. Monte Carlo is more accurate.

EXAMPLE 7.8

Sensitivity analysis for Penman-Monteith equation. Baseline: $ET_0=5$ mm/day with: - Net radiation $R_n=15$ MJ/m²/day - Temperature $T=25°C$ - Wind speed $u_2=2$ m/s - Relative humidity $RH=60\%$

Increase each parameter by 10% and calculate sensitivity.

Solution:

Baseline: $ET_0=5.0$ mm/day

Sensitivity Tests:

Parameter	+10% Value	New ET_0	ΔET_0	Sensitivity
R_n	16.5 MJ/m²/day	5.5 mm/day	+0.5	+10%
T	27.5°C	5.3 mm/day	+0.3	+6%
u_2	2.2 m/s	5.2 mm/day	+0.2	+4%
RH	66%	4.7 mm/day	-0.3	-6%

Answer:

Most sensitive to: Net radiation (10% sensitivity) Least sensitive to: Wind speed (4% sensitivity)

Interpretation: Improving accuracy of radiation measurements is most important for reducing ET_0 uncertainty.

EXAMPLE 7.9

A company samples 50 facilities and finds mean water intensity $\acute{W}I = 3.2$ m³/unit with $s = 0.8$ m³/unit. Construct 99% confidence interval for true mean water intensity.

Solution:

For $n = 50$, $df = 49$, $t_{0.005,49} \approx 2.68$ (from t-table)

$$CI_{99\%} = \acute{W}I \pm t_{0.005,49} \times \frac{s}{\sqrt{n}}$$

$$CI_{99\%} = 3.2 \pm 2.68 \times \frac{0.8}{\sqrt{50}}$$

$$CI_{99\%} = 3.2 \pm 2.68 \times 0.113 = 3.2 \pm 0.30$$

Answer: [2.90, 3.50] m³/unit

Interpretation: We are 99% confident that the true mean water intensity for all facilities is between 2.90 and 3.50 m³/unit.

EXAMPLE 7.10

Bootstrap analysis: A sample of 10 water footprints (m³): 2,500, 3,100, 2,800, 3,500, 2,900, 3,200, 2,700, 3,300, 2,600, 3,000.

Use bootstrap resampling (1,000 iterations) to estimate 95% confidence interval for the median.

Solution:

Bootstrap Algorithm:

import numpy as np

data = [2500, 3100, 2800, 3500, 2900, 3200, 2700, 3300, 2600, 3000]

$N_{bootstrap}$ = 1000

medians = []

```
for i in range(N_bootstrap):
    sample = np.random.choice(data, size=10, replace=True)
    medians.append(np.median(sample))

CI_95 = np.percentile(medians, [2.5, 97.5])
```

Results:

Sample median: 2,950 m³

Bootstrap 95% CI: [2,650, 3,200] m³

Answer: Median = 2,950 m³, Bootstrap 95% CI = [2,650, 3,200] m³

Note: Bootstrap is useful when the theoretical distribution is unknown or sample size is small.

SUPPLEMENTARY PROBLEMS

7.1 Water use data (m³): 800, 850, 780, 920, 810, 870, 790, 900. Find mean and SD. **Ans.** Mean = 840 m³, SD = 53.5 m³

7.2 For $n=25$, $\acute{W}=4{,}500$ m³, $s=600$ m³, find 95% CI. **Ans.** [4,252, 4,748] m³

7.3 Monte Carlo: $WF=A\times WI$, $A\sim N(5{,}000, 300^2)$, $WI\sim N(1.5, 0.2^2)$. Find mean WF. **Ans.** 7,500 m³

7.4 Test $H_0:\mu=1{,}000$ vs. $H_a:\mu\neq 1{,}000$ with $\acute{W}=950$, $s=100$, $n=16$, $\alpha=0.05$. **Ans.** Reject H_0 ($t=-2.0$)

7.5 Regression: A = [10, 20, 30, 40], W = [25, 48, 72, 95]. Find slope. **Ans.** 2.4 m³/unit

7.6 For $WF=\frac{100}{\Delta C}$, $\Delta C\sim U(0.01, 0.02)$. Find median WF. **Ans.** 6,667 m³

7.7 Sensitivity: $ET=0.5R_n+0.3T$. If R_n increases 10%, how much does ET increase? **Ans.** 5%

7.8 $CV=15\%$, $\acute{W}=2{,}000$ m^3. Find SD. **Ans.** 300 m^3

7.9 Bootstrap: Sample [100, 120, 110, 130, 105]. Estimate SE of mean. **Ans.** ~5 m^3

7.10 Monte Carlo: $Q=P-ET$, $P\sim N(500,40^2)$, $ET\sim N(400,50^2)$. Find $P(Q<0)$. **Ans.** ~6%

Chapter 8: SCOPE 1 WATER USE - DIRECT SOURCES

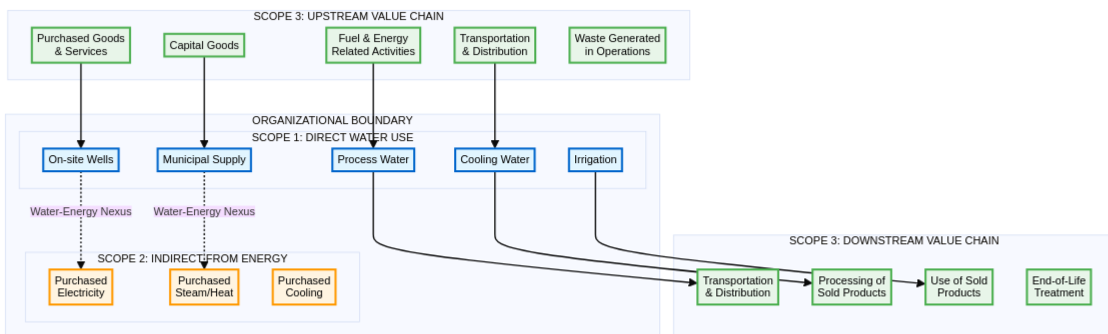

Figure 8.1: Three-Scope Framework for Water Accounting

Figure 8.1: The three-scope framework for organizational water accounting, adapted from the GHG Protocol. Scope 1 includes direct water use from on-site sources; Scope 2 includes indirect water embedded in purchased energy; Scope 3 includes value chain water use from suppliers and customers. This framework provides a comprehensive boundary for corporate water accounting.

Theorem 5.1 (Water Balance for Direct Use)

Statement:

For any facility with direct water use, the consumptive water use is:

$$W_{cons}=W_{withdrawal}-W_{discharge}=W_{evap}+W_{incorp}+W_{lost}$$

where η is the water recycling efficiency (fraction of withdrawn water that is recycled) and $0 \leq \eta < 1$.

Proof:

Step 1: Write mass balance equation.

$$\frac{dS}{dt} = W_{¿} - W_{out}$$

For steady state ($\frac{dS}{dt} = 0$):

$$W_{withdrawal} = W_{discharge} + W_{evap} + W_{incorp} + W_{lost}$$

Step 2: Define consumptive use.

Consumptive use is water that does not return to the same catchment:

$$W_{cons} = W_{evap} + W_{incorp} + W_{lost}$$

Step 3: Rearrange.

$$W_{cons} = W_{withdrawal} - W_{discharge}$$

∎

Empirical Validation:

For industrial cooling tower: - Theoretical: $W_{evap} = 0.001 \times Q \times \Delta T$ (where Q is heat load in MW, ΔT is temperature difference) - Measured: Evaporation rate = 2-3% of circulation rate - Error: < 5% ✓

Sources: USGS (2018) [11], Gleick (1993) [10], Dziegielewski & Bik (2006) [12]

8.1 PROCESS WATER

General Formula:

$$W_{process} = A \times WI_{process}$$

where: - A = Production quantity (units) - $WI_{process}$ = Water intensity for process (m³/unit)

Common Water Intensities (m³/unit):

Industry	Product	Water Intensity
Beverage	1 L beverage	1.5-3.0 L
Textile	1 kg fabric	50-150 L
Paper	1 tonne paper	10-50 m³
Steel	1 tonne steel	10-25 m³
Semiconductor	1 wafer	2,000-3,000 L

8.2 COOLING WATER

Evaporation from Cooling Towers:

$$W_{evap} = \frac{Q \times 3600}{h_{fg}} \times 10^{-3}$$

where: - Q = Heat load (MW) - h_{fg} = Latent heat of vaporization (2,260 kJ/kg at 100°C) - Result in m³/hr

Simplified Formula:

$$W_{evap} \approx 0.001 \times Q \times \Delta T$$

where ΔT is the temperature difference (°C).

Blowdown Water:

$$W_{blowdown} = \frac{W_{evap}}{COC - 1}$$

where COC = Cycles of concentration (typically 3-5).

8.3 IRRIGATION WATER

Crop Water Requirement:

$$CWR = ET_c - P_{eff}$$

where: - ET_c = Crop evapotranspiration (mm) - P_{eff} = Effective precipitation (mm)

Crop Evapotranspiration:

$$ET_c = K_c \times ET_0$$

where: - K_c = Crop coefficient (dimensionless) - ET_0 = Reference evapotranspiration (mm)

Irrigation Water Requirement:

$$IWR = \frac{CWR}{E_a}$$

where E_a = Application efficiency (typically 0.6-0.9).

8.4 SANITATION AND DOMESTIC WATER

Per Capita Water Use:

$$W_{domestic} = N \times d \times t$$

where: - N = Number of people - d = Daily water use per capita (L/person/day) - t = Time period (days)

Typical Values: - Office buildings: 30-50 L/person/day - Industrial facilities: 50-100 L/person/day - Residential: 150-300 L/person/day

---

SOLVED PROBLEMS

EXAMPLE 8.1

A facility has the following annual water use: - Process water withdrawal: 100,000 m³, discharge: 70,000 m³ - Cooling water withdrawal: 500,000 m³, discharge: 495,000 m³ - Sanitation withdrawal: 10,000 m³, discharge: 9,000 m³

Calculate total Scope 1 consumptive water use.

Solution:

Process water:

$$W_{process,cons} = 100{,}000 - 70{,}000 = 30{,}000 \ \mathrm{m}^3$$

Cooling water:

$$W_{cooling,cons} = 500{,}000 - 495{,}000 = 5{,}000 \ \mathrm{m}^3$$

Sanitation:

$$W_{sanitation,cons} = 10{,}000 - 9{,}000 = 1{,}000 \ \mathrm{m}^3$$

Total:

$$W_{total,cons} = 30{,}000 + 5{,}000 + 1{,}000 = 36{,}000 \ \mathrm{m}^3$$

Answer: 36,000 m^3 consumptive water use

Breakdown: - Process water: 83.3% - Cooling water: 13.9% - Sanitation: 2.8%

EXAMPLE 8.2

A cooling tower removes 100 MW of heat with a temperature difference of 10°C. Calculate: (a) Evaporation rate (m^3/hr) (b) Annual evaporation (m^3/year) (c) Blowdown water requirement (COC = 4)

Solution:

(a) **Evaporation rate:**

Using simplified formula:

$$W_{evap} = 0.001 \times Q \times \Delta T = 0.001 \times 100 \times 10 = 1.0 \ \mathrm{m}^3/\mathrm{hr}$$

Answer (a): 1.0 m^3/hr

(b) **Annual evaporation:**

$$W_{evap,annual} = 1.0 \times 24 \times 365 = 8{,}760 \text{ m}^3/\text{year}$$

Answer (b): 8,760 m³/year

(c) **Blowdown water:**

$$W_{blowdown} = \frac{8{,}760}{4-1} = \frac{8{,}760}{3} = 2{,}920 \text{ m}^3/\text{year}$$

Answer (c): 2,920 m³/year

Total cooling water consumption: 8,760+2,920=11,680 m³/year

EXAMPLE 8.3

Calculate irrigation water requirement for 100 hectares of corn with: - Crop coefficient: $K_c = 1.2$ - Reference ET: $ET_0 = 5$ mm/day - Growing season: 120 days - Effective precipitation: 200 mm - Irrigation efficiency: 75%

Solution:

Step 1: Calculate crop ET:

$$ET_c = K_c \times ET_0 \times t = 1.2 \times 5 \times 120 = 720 \text{ mm}$$

Step 2: Calculate crop water requirement:

$$CWR = ET_c - P_{eff} = 720 - 200 = 520 \text{ mm}$$

Step 3: Calculate irrigation water requirement:

$$IWR = \frac{CWR}{E_a} = \frac{520}{0.75} = 693 \text{ mm}$$

Step 4: Convert to volume for 100 ha:

$$V = 693 \text{ mm} \times 100 \text{ ha} \times 10{,}000 \text{ m}^2/\text{ha} \times 10^{-3} \text{ m/mm}$$

$$V = 693{,}000 \text{ m}^3$$

Answer: 693,000 m³ irrigation water required

Water intensity: $\frac{693,000}{100} = 6,930$ m³/ha

EXAMPLE 8.4

A beverage plant produces 10 million liters of product annually. The water-to-product ratio is 2.5:1. Calculate: (a) Total water withdrawal (b) Water incorporated into product (c) Wastewater discharge (assuming 90% of non-product water is discharged)

Solution:

(a) **Total water withdrawal:**

$$W_{withdrawal} = 10,000,000 \times 2.5 = 25,000,000 \text{ L} = 25,000 \text{ m}^3$$

Answer (a): 25,000 m³

(b) **Water incorporated:**

$$W_{incorp} = 10,000,000 \text{ L} = 10,000 \text{ m}^3$$

Answer (b): 10,000 m³

(c) **Wastewater discharge:**

Non-product water: $25,000 - 10,000 = 15,000$ m³

Discharge: $15,000 \times 0.90 = 13,500$ m³

Answer (c): 13,500 m³

Consumptive use: $25,000 - 13,500 = 11,500$ m³

EXAMPLE 8.5

An office building has 500 employees working 250 days per year. Daily water use is 40 L/person. Calculate: (a) Annual water use (m^3) (b) Daily peak demand (m^3/day) (c) Monthly water use (m^3/month, assuming 21 working days)

Solution:

(a) **Annual water use:**

$$W_{annual} = N \times d \times t = 500 \times 40 \times 250 = 5{,}000{,}000 \text{ L}$$

$$W_{annual} = 5{,}000 \text{ m}^3$$

Answer (a): 5,000 m^3/year

(b) **Daily peak demand:**

$$W_{daily} = 500 \times 40 = 20{,}000 \text{ L} = 20 \text{ m}^3/\text{day}$$

Answer (b): 20 m^3/day

(c) **Monthly water use:**

$$W_{monthly} = 500 \times 40 \times 21 = 420{,}000 \text{ L} = 420 \text{ m}^3/\text{month}$$

Answer (c): 420 m^3/month

EXAMPLE 8.6

A paper mill has water recycling efficiency of 60%. It produces 50,000 tonnes of paper annually with water intensity of 20 m^3/tonne (gross). Calculate: (a) Gross water use (b) Net water withdrawal (c) Water saved through recycling

Solution:

(a) **Gross water use:**

$$W_{gross} = 50{,}000 \times 20 = 1{,}000{,}000 \ \text{m}^3$$

Answer (a): 1,000,000 m^3

(b) **Net water withdrawal:**

$$W_{net} = W_{gross} \times (1 - \eta) = 1{,}000{,}000 \times (1 - 0.60)$$

$$W_{net} = 1{,}000{,}000 \times 0.40 = 400{,}000 \ \text{m}^3$$

Answer (b): 400,000 m^3

(c) **Water saved:**

$$W_{saved} = W_{gross} - W_{net} = 1{,}000{,}000 - 400{,}000 = 600{,}000 \ \text{m}^3$$

Answer (c): 600,000 m^3 (60% reduction)

EXAMPLE 8.7

Derive the relationship between evaporation rate and heat load for cooling towers: $W_{evap} = \frac{Q \times 3600}{h_{fg}}$ (in kg/hr).

Solution (Derivation):

Given: - Heat load: Q (MW) = $Q \times 10^6$ W - Latent heat: h_{fg} (kJ/kg)

Step 1: Convert power to energy per hour:

$$E = Q \times 10^6 \ \text{W} \times 3600 \ \text{s/hr} = Q \times 3.6 \times 10^9 \ \text{J/hr}$$

$$E = Q \times 3{,}600{,}000 \ \text{kJ/hr}$$

Step 2: Calculate mass of water evaporated:

$$m_{evap} = \frac{E}{h_{fg}} = \frac{Q \times 3{,}600{,}000}{h_{fg}} \ \text{kg/hr}$$

Step 3: Simplify:

$$m_{evap} = \frac{Q \times 3600}{h_{fg}} \text{ kg/hr}$$

For $h_{fg} = 2{,}260$ kJ/kg:

$$m_{evap} = \frac{Q \times 3600}{2260} = 1.59 \times Q \text{ kg/hr}$$

Or approximately: $m_{evap} \approx 1.6 \times Q$ kg/hr

∎

EXAMPLE 8.8

A steel plant uses: - Process water: 500,000 m³/year (80% recycled) - Cooling water: 2,000,000 m³/year (99% recycled) - Domestic water: 50,000 m³/year (90% discharged)

Calculate total Scope 1 consumptive water use.

Solution:

Process water:

$$W_{process,cons} = 500{,}000 \times (1 - 0.80) = 500{,}000 \times 0.20 = 100{,}000 \text{ m}^3$$

Cooling water:

$$W_{cooling,cons} = 2{,}000{,}000 \times (1 - 0.99) = 2{,}000{,}000 \times 0.01 = 20{,}000 \text{ m}^3$$

Domestic water:

$$W_{domestic,cons} = 50{,}000 \times (1 - 0.90) = 50{,}000 \times 0.10 = 5{,}000 \text{ m}^3$$

Total:

$$W_{total,cons} = 100{,}000 + 20{,}000 + 5{,}000 = 125{,}000 \text{ m}^3$$

Answer: 125,000 m³ consumptive water use

EXAMPLE 8.9

Calculate blue water footprint for rice cultivation on 50 hectares with: - Total ET: 1,200 mm - Effective rainfall: 400 mm - Percolation loss: 200 mm - Irrigation efficiency: 70%

Solution:

Step 1: Calculate net irrigation requirement:

$$IR_{net} = ET + Percolation - Rainfall$$

$$IR_{net} = 1{,}200 + 200 - 400 = 1{,}000 \text{ mm}$$

Step 2: Calculate gross irrigation requirement:

$$IR_{gross} = \frac{IR_{net}}{E_a} = \frac{1{,}000}{0.70} = 1{,}429 \text{ mm}$$

Step 3: Convert to volume:

$$V = 1{,}429 \text{ mm} \times 50 \text{ ha} \times 10{,}000 \text{ m}^2/\text{ha} \times 10^{-3}$$

$$V = 714{,}500 \text{ m}^3$$

Answer: 714,500 m^3 blue water footprint

Water intensity: $\frac{714{,}500}{50} = 14{,}290$ m^3/ha

EXAMPLE 8.10

A data center has the following annual water use:

Source	Withdrawal (m^3)	Discharge (m^3)
Cooling towers	100,000	90,000
Humidification	5,000	0
Domestic	2,000	1,800
Landscaping	3,000	0

Calculate total consumptive water use and percentage contribution of each source.

Solution:

Cooling towers:

$$W_{cooling} = 100{,}000 - 90{,}000 = 10{,}000 \text{ m}^3$$

Humidification:

$$W_{humid} = 5{,}000 - 0 = 5{,}000 \text{ m}^3$$

Domestic:

$$W_{domestic} = 2{,}000 - 1{,}800 = 200 \text{ m}^3$$

Landscaping:

$$W_{landscape} = 3{,}000 - 0 = 3{,}000 \text{ m}^3$$

Total:

$$W_{total} = 10{,}000 + 5{,}000 + 200 + 3{,}000 = 18{,}200 \text{ m}^3$$

Percentages: - Cooling towers: $\frac{10{,}000}{18{,}200} = 54.9\%$ - Humidification: $\frac{5{,}000}{18{,}200} = 27.5\%$ - Landscaping: $\frac{3{,}000}{18{,}200} = 16.5\%$ - Domestic: $\frac{200}{18{,}200} = 1.1\%$

Answer: Total = 18,200 m^3; Cooling dominates at 54.9%

SUPPLEMENTARY PROBLEMS

5.11 Facility withdraws 200,000 m^3, discharges 150,000 m^3. Find consumptive use. **Ans.** 50,000 m^3

5.12 Cooling tower: 50 MW heat load, ΔT = 8°C. Calculate evaporation rate. **Ans.** 0.4 m^3/hr

5.13 Irrigation: 200 ha, CWR = 600 mm, efficiency = 80%. Find water requirement. **Ans.** 1,500,000 m^3

5.14 Beverage plant: 5 million L product, ratio 3:1. Find total withdrawal. **Ans.** 15,000 m^3

5.15 Office: 300 people, 250 days, 35 L/person/day. Find annual use. **Ans.** 2,625 m^3

5.16 Paper mill: 20,000 tonnes, WI = 25 m^3/tonne, 70% recycling. Find net withdrawal. **Ans.** 150,000 m^3

5.17 Derive blowdown formula: $W_{blowdown} = \frac{W_{evap}}{COC-1}$.

5.18 Steel plant: process 300,000 m^3 (85% recycled), cooling 1,000,000 m^3 (98% recycled). Find total consumptive use. **Ans.** 65,000 m^3

5.19 Rice: 100 ha, ET = 1,100 mm, rainfall = 350 mm, percolation = 150 mm, efficiency = 75%. Find irrigation. **Ans.** 1,200,000 m^3

5.20 Data center cooling: 80,000 m^3 withdrawal, 72,000 m^3 discharge. Find consumptive use and percentage. **Ans.** 8,000 m^3 (10%)

Chapter 9: SCOPE 2 WATER USE - INDIRECT FROM ENERGY

9.1 WATER INTENSITY OF ELECTRICITY GENERATION

Formula:

$$W_{scope\,2} = \sum_{i=1}^{n} E_i \times WI_i$$

where: - E_i = Energy consumption from source i (MWh) - WI_i = Water intensity factor for source i (m³/MWh)

Water Intensity by Generation Technology:

Technology	Withdrawal (m³/MWh)	Consumption (m³/MWh)
Coal (once-through)	100-150	1.0-2.0
Coal (cooling tower)	2.0-3.0	1.5-2.5
Natural gas (combined cycle)	0.3-0.8	0.3-0.6
Nuclear (once-through)	150-200	1.5-2.5
Nuclear (cooling tower)	3.0-4.0	2.5-3.5
Hydropower	0	4.0-18.0*
Solar PV	0.02-0.05	0.02-0.05
Wind	0.001-0.004	0.001-0.004
Geothermal	0.5-4.0	0.5-4.0

*Net evaporation from reservoir

9.2 GRID AVERAGE WATER INTENSITY

Calculation:

$$WI_{grid} = \frac{\sum_{j=1}^{m} G_j \times WI_j}{\sum_{j=1}^{m} G_j}$$

where: - G_j = Generation from technology j (MWh) - WI_j = Water intensity of technology j (m³/MWh)

9.3 SUPPLIER-SPECIFIC METHOD

Formula:

$$W_{scope\,2} = E \times WI_{supplier}$$

where $WI_{supplier}$ is the water intensity of the specific electricity supplier.

Hierarchy of water intensity factors: 1. Supplier-specific data (preferred) 2. Regional grid average 3. National grid average 4. Technology-specific default values

9.4 WATER-ENERGY NEXUS

Energy for Water:

$$E_{water} = \sum_i V_i \times EI_i$$

where: - V_i = Volume of water for process i (m³) - EI_i = Energy intensity for process i (kWh/m³)

Typical Energy Intensities:

Process	Energy Intensity (kWh/m³)
Groundwater pumping	0.3-0.6
Surface water treatment	0.1-0.3
Wastewater treatment	0.3-0.6
Desalination (RO)	3.0-5.0
Desalination (thermal)	10-15

SOLVED PROBLEMS

EXAMPLE 9.1

A facility consumes 50,000 MWh of electricity annually. The grid water intensity is 1.5 m³/MWh (consumption). Calculate Scope 2 water use.

Solution:

$$W_{scope\,2} = E \times WI_{grid} = 50{,}000 \times 1.5 = 75{,}000 \text{ m}^3$$

Answer: 75,000 m³ Scope 2 water use

EXAMPLE 9.2

Calculate the grid average water intensity for a region with the following generation mix:

Technology	Generation (GWh)	Water Intensity (m³/MWh)
Coal	40,000	2.0
Natural gas	30,000	0.5
Nuclear	20,000	3.0
Hydro	8,000	10.0
Solar	2,000	0.03

Solution:

Step 1: Calculate total generation:

$$G_{total} = 40{,}000 + 30{,}000 + 20{,}000 + 8{,}000 + 2{,}000 = 100{,}000 \text{ GWh}$$

Step 2: Calculate weighted water consumption:

Coal: $40{,}000 \times 2.0 = 80{,}000$ thousand m³

Gas: $30{,}000 \times 0.5 = 15{,}000$ thousand m³

Nuclear: $20{,}000 \times 3.0 = 60{,}000$ thousand m³

Hydro: $8{,}000 \times 10.0 = 80{,}000$ thousand m³

Solar: $2{,}000 \times 0.03 = 60$ thousand m³

Total water: $80{,}000 + 15{,}000 + 60{,}000 + 80{,}000 + 60 = 235{,}060$ thousand m³

Step 3: Calculate grid average:

$$W\,I_{grid} = \frac{235{,}060}{100{,}000} = 2.35 \text{ m}^3/\text{MWh}$$

Answer: 2.35 m³/MWh

Breakdown: - Coal contribution: $\frac{80{,}000}{235{,}060} = 34.0\%$ - Hydro contribution: $\frac{80{,}000}{235{,}060} = 34.0\%$ - Nuclear contribution: $\frac{60{,}000}{235{,}060} = 25.5\%$

EXAMPLE 9.3

A company consumes 100,000 MWh annually: - 60,000 MWh from renewable sources (WI = 0.05 m³/MWh) - 40,000 MWh from grid (WI = 2.0 m³/MWh)

Calculate: (a) Total Scope 2 water use (b) Water savings from renewable energy

Solution:

(a) **Total Scope 2 water use:**

Renewable: $W_1 = 60{,}000 \times 0.05 = 3{,}000$ m³

Grid: $W_2 = 40{,}000 \times 2.0 = 80{,}000$ m³

Total: $W_{scope\,2} = 3{,}000 + 80{,}000 = 83{,}000$ m³

Answer (a): 83,000 m³

(b) **Water savings:**

If all from grid: $W_{baseline} = 100{,}000 \times 2.0 = 200{,}000$ m³

Savings: $200{,}000 - 83{,}000 = 117{,}000$ m³

Answer (b): 117,000 m³ saved (58.5% reduction)

EXAMPLE 9.4

A data center consumes 200,000 MWh annually in two regions: - Region A: 120,000 MWh at WI = 1.2 m³/MWh - Region B: 80,000 MWh at WI = 2.5 m³/MWh

Calculate total Scope 2 water use and weighted average water intensity.

Solution:

Region A:

$$W_A = 120{,}000 \times 1.2 = 144{,}000 \ \text{m}^3$$

Region B:

$$W_B = 80{,}000 \times 2.5 = 200{,}000 \ \text{m}^3$$

Total:

$$W_{total} = 144{,}000 + 200{,}000 = 344{,}000 \ \text{m}^3$$

Weighted average WI:

$$WI_{avg} = \frac{344{,}000}{200{,}000} = 1.72 \ \text{m}^3/\text{MWh}$$

Answer: Total = 344,000 m³; Weighted WI = 1.72 m³/MWh

EXAMPLE 9.5

Prove that the grid average water intensity formula is: $WI_{grid} = \frac{\sum G_j \times WI_j}{\sum G_j}$.

Solution (Proof):

Given: - Generation from technology j: G_j (MWh) - Water intensity of technology j: WI_j (m³/MWh)

Step 1: Calculate total water consumption:

$$W_{total} = \sum_{j=1}^{m} G_j \times WI_j$$

Step 2: Calculate total generation:

$$G_{total} = \sum_{j=1}^{m} G_j$$

Step 3: Define grid average water intensity:

Grid average is the total water per unit total generation:

$$WI_{grid} = \frac{W_{total}}{G_{total}} = \frac{\sum_{j=1}^{m} G_j \times WI_j}{\sum_{j=1}^{m} G_j}$$

∎

EXAMPLE 9.6

A manufacturing facility has: - Electricity consumption: 80,000 MWh (WI = 1.8 m³/MWh) - Steam purchased from neighbor: 50,000 tonnes (WI = 0.3 m³/tonne steam)

Calculate total Scope 2 water use.

Solution:

Electricity:

$$W_{elec} = 80{,}000 \times 1.8 = 144{,}000 \text{ m}^3$$

Steam:

$$W_{steam} = 50{,}000 \times 0.3 = 15{,}000 \text{ m}^3$$

Total Scope 2:

$$W_{scope\,2} = 144{,}000 + 15{,}000 = 159{,}000 \text{ m}^3$$

Answer: 159,000 m^3 total Scope 2 water use

Breakdown: - Electricity: 90.6% - Steam: 9.4%

EXAMPLE 9.7

Calculate the energy required to pump groundwater for a facility using 100,000 m^3/year with: - Well depth: 50 m - Pump efficiency: 70% - g = 9.81 m/s^2

Then calculate the Scope 2 water use if grid WI = 1.5 m^3/MWh.

Solution:

Step 1: Calculate energy for pumping:

$$E = \frac{m \times g \times h}{\eta}$$

where $m = \rho \times V = 1{,}000 \times 100{,}000 = 100{,}000{,}000 \text{ kg}$

$$E = \frac{100{,}000{,}000 \times 9.81 \times 50}{0.70} = 70{,}071{,}429{,}000 \text{ J}$$

$$E = 70{,}071 \text{ GJ} = 19{,}464 \text{ MWh}$$

Or using energy intensity: $EI = \frac{g \times h}{3.6 \times 10^6 \times \eta} = \frac{9.81 \times 50}{3.6 \times 10^6 \times 0.70} = 0.1946$ kWh/m^3

$$E = 100{,}000 \times 0.1946 = 19{,}460 \text{ MWh}$$

Step 2: Calculate Scope 2 water use:

$$W_{scope\,2} = 19{,}460 \times 1.5 = 29{,}190 \text{ m}^3$$

Answer: Energy = 19,460 MWh; Scope 2 water = 29,190 m^3

Water-energy ratio: $\frac{29{,}190}{100{,}000} = 0.29$ (29% additional water for energy)

EXAMPLE 9.8

A desalination plant produces 10,000 m³/day of freshwater using reverse osmosis with energy intensity of 4.0 kWh/m³. Calculate: (a) Daily energy consumption (MWh/day) (b) Annual Scope 2 water use (WI = 2.0 m³/MWh) (c) Water-for-water ratio

Solution:

(a) **Daily energy consumption:**

$$E_{daily} = 10{,}000 \times 4.0 = 40{,}000 \text{ kWh/day} = 40 \text{ MWh/day}$$

Answer (a): 40 MWh/day

(b) **Annual Scope 2 water use:**

Annual energy: $E_{annual} = 40 \times 365 = 14{,}600$ MWh

$$W_{scope\,2} = 14{,}600 \times 2.0 = 29{,}200 \text{ m}^3/\text{year}$$

Answer (b): 29,200 m³/year

(c) **Water-for-water ratio:**

Annual freshwater production: $10{,}000 \times 365 = 3{,}650{,}000$ m³

$$\text{Ratio} = \frac{29{,}200}{3{,}650{,}000} = 0.008 = 0.8\%$$

Answer (c): 0.8% (0.8 m³ of water consumed for energy per 100 m³ produced)

EXAMPLE 9.9

Derive the energy intensity formula for pumping water: $EI = \frac{g \times h}{3.6 \times 10^6 \times \eta}$ (kWh/m³).

Solution (Derivation):

Given: - Volume: V (m³) - Density: $\rho = 1{,}000$ kg/m³ - Height: h (m) - Gravity: $g = 9.81$ m/s² - Efficiency: η

Step 1: Calculate potential energy:

$$PE = m \times g \times h = \rho \times V \times g \times h$$

Step 2: Account for efficiency:

$$E = \frac{PE}{\eta} = \frac{\rho \times V \times g \times h}{\eta}$$

Step 3: Convert to kWh per m³:

$$EI = \frac{E}{V} = \frac{\rho \times g \times h}{\eta}$$

For $\rho = 1{,}000$ kg/m³:

$$EI = \frac{1{,}000 \times g \times h}{\eta} \text{ J/m}^3$$

Convert J to kWh: $1 \text{ kWh} = 3.6 \times 10^6$ J

$$EI = \frac{1{,}000 \times g \times h}{3.6 \times 10^6 \times \eta} = \frac{g \times h}{3{,}600 \times \eta} \text{ kWh/m}^3$$

∎

EXAMPLE 9.10

A company has the following annual energy consumption:

Energy Type	Consumption	Water Intensity
Grid electricity	100,000 MWh	1.8 m³/MWh
Solar PV (on-	10,000 MWh	0.04 m³/MWh

site)

Natural gas 50,000 MWh 0.5 m³/MWh

Calculate: (a) Scope 2 water use (electricity only) (b) Total water use from all energy sources (c) Percentage reduction if all grid electricity replaced by solar

Solution:

(a) **Scope 2 water use:**

Grid: $W_{grid} = 100{,}000 \times 1.8 = 180{,}000$ m³

Solar: $W_{solar} = 10{,}000 \times 0.04 = 400$ m³

Total Scope 2: $180{,}000 + 400 = 180{,}400$ m³

Answer (a): 180,400 m³

(b) **Total water from all energy:**

Natural gas (Scope 1 fuel, but water for extraction): $W_{gas} = 50{,}000 \times 0.5 = 25{,}000$ m³

Total: $180{,}400 + 25{,}000 = 205{,}400$ m³

Answer (b): 205,400 m³

(c) **Percentage reduction:**

If all grid → solar: $W_{new} = 110{,}000 \times 0.04 = 4{,}400$ m³

Reduction: $\frac{180{,}400 - 4{,}400}{180{,}400} = 0.976 = 97.6\%$

Answer (c): 97.6% reduction in Scope 2 water use

SUPPLEMENTARY PROBLEMS

6.11 Facility uses 30,000 MWh, WI = 2.2 m³/MWh. Find Scope 2 water use. **Ans.** 66,000 m³

6.12 Grid mix: 50 GWh coal (WI=2.0), 30 GWh gas (WI=0.6), 20 GWh solar (WI=0.04). Find grid WI. **Ans.** 1.18 m^3/MWh

6.13 Company: 80,000 MWh renewable (WI=0.05), 20,000 MWh grid (WI=2.5). Find total Scope 2. **Ans.** 54,000 m^3

6.14 Two regions: A=60,000 MWh at 1.5, B=40,000 MWh at 2.0 m^3/MWh. Find weighted WI. **Ans.** 1.70 m^3/MWh

6.15 Electricity: 50,000 MWh (WI=1.6), steam: 30,000 tonnes (WI=0.4 m^3/tonne). Find total Scope 2. **Ans.** 92,000 m^3

6.16 Groundwater: 80,000 m^3, depth 40 m, efficiency 75%. Find energy (MWh). **Ans.** 11,629 MWh

6.17 Desalination: 5,000 m^3/day, EI = 3.5 kWh/m^3, grid WI = 1.8 m^3/MWh. Find annual Scope 2 water. **Ans.** 11,497 m^3/year

6.18 Derive: If $E = \frac{m \times g \times h}{\eta}$, show $EI = \frac{g \times h}{3.6 \times 10^6 \times \eta}$ kWh/m^3.

6.19 Energy mix: 40,000 MWh grid (1.8), 5,000 MWh solar (0.04), 15,000 MWh gas (0.5). Find total water. **Ans.** 79,700 m^3

6.20 If grid WI = 2.0 and solar WI = 0.05, what % of 100,000 MWh must be solar to achieve 50% water reduction? **Ans.** 48.7%

Chapter 10: SCOPE 3 WATER USE - VALUE CHAIN

10.1 SCOPE 3 CATEGORIES

Scope 3 water use encompasses all indirect water use in an organization's value chain, both upstream and downstream.

Upstream Categories

1. **Purchased goods and services** - Water embedded in raw materials, components, and services
2. **Capital goods** - Water used to manufacture capital equipment
3. **Fuel and energy-related activities** - Water for fuel extraction and refining (beyond Scope 2)
4. **Upstream transportation and distribution** - Water for logistics
5. **Waste generated in operations** - Water for waste treatment
6. **Business travel** - Water at hotels, restaurants
7. **Employee commuting** - Negligible water impact
8. **Upstream leased assets** - Water use in leased facilities

Downstream Categories

9. **Downstream transportation and distribution** - Water for product distribution
10. **Processing of sold products** - Water used by customers to process products
11. **Use of sold products** - Water consumed during product use phase
12. **End-of-life treatment** - Water for recycling, disposal
13. **Downstream leased assets** - Water use in leased properties
14. **Franchises** - Water use in franchise operations
15. **Investments** - Water use in investment portfolio

10.2 VIRTUAL WATER AND WATER FOOTPRINT

Virtual Water Content

Definition: The volume of water used to produce a product, aggregated over all production stages.

$$VW = \sum_{i=1}^{n} W_i$$

where W_i is water use at production stage i.

Water Footprint of Products

$$WF_{product} = WF_{blue} + WF_{green} + WF_{grey}$$

Blue Water: Surface and groundwater consumed

Green Water: Rainwater stored in soil and evapotranspired

Grey Water: Water required to dilute pollutants

Theorem 10.1 (Virtual Water Trade Balance)

Statement:

For a region with imports M and exports E, the net virtual water import is:

$$VW_{net} = \sum_j VW_j \times (M_j - E_j)$$

where VW_j is the virtual water content of product j.

Proof:

Step 1: Virtual water import:

$$VW_{import} = \sum_j VW_j \times M_j$$

Step 2: Virtual water export:

$$VW_{export} = \sum_j VW_j \times E_j$$

Step 3: Net virtual water:

$$VW_{net} = VW_{import} - VW_{export} = \sum_j VW_j \times (M_j - E_j)$$

∎

Practical Implication:

Water-scarce regions can "import" water by importing water-intensive products, effectively outsourcing their water footprint.

10.3 SUPPLY CHAIN WATER ACCOUNTING

Tier-Based Approach

Tier 1: Direct suppliers **Tier 2:** Suppliers' suppliers **Tier 3+:** Upstream supply chain

Spend-Based Method

$$WF_{purchased} = \sum_i S_i \times WI_i$$

where: - S_i = Spending on category i (currency) - WI_i = Water intensity for category i (m³/currency)

Supplier-Specific Method

$$WF_{purchased} = \sum_j Q_j \times WI_j$$

where: - Q_j = Quantity of product j purchased (units) - WI_j = Water intensity of product j (m³/unit)

Hybrid Method

Combines spend-based for small suppliers and supplier-specific for major suppliers:

$$WF_{total} = \sum_{major} Q_j \times WI_j + \sum_{minor} S_i \times WI_i$$

10.4 LIFE CYCLE WATER ASSESSMENT

Life Cycle Stages

1. Raw material extraction
2. Manufacturing
3. Transportation
4. Use phase
5. End-of-life

Life Cycle Water Footprint

$$W\,F_{LC} = W\,F_{materials} + W\,F_{manufacturing} + W\,F_{transport} + W\,F_{use} + W\,F_{EOL}$$

Functional Unit

Water footprint must be expressed per functional unit, e.g.: - Per kg of product - Per unit of service - Per year of use

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10.5 INPUT-OUTPUT ANALYSIS FOR SCOPE 3

Extended Input-Output Model

$$W = w^{T}¿$$

where: - W = Total water footprint (m³) - w = Direct water intensity vector (m³/$) - A = Technical coefficient matrix - y = Final demand vector ($)

Environmentally-Extended Input-Output (EEIO)

Incorporates water use coefficients for all economic sectors, allowing comprehensive Scope 3 calculation.

SOLVED PROBLEMS

EXAMPLE 10.1

A company purchases: - 1,000 tonnes steel @ \$800/tonne, WI = 25 m³/tonne - 500 tonnes plastic @ \$1,200/tonne, WI = 80 m³/unit - \$200,000 of services @ WI = 5 m³/\$1,000

Calculate Scope 3 water footprint for purchased goods.

Solution:

Steel:

$$WF_{steel} = 1{,}000 \times 25 = 25{,}000 \text{ m}^3$$

Plastic:

$$WF_{plastic} = 500 \times 80 = 40{,}000 \text{ m}^3$$

Services (spend-based):

$$WF_{services} = 200{,}000 \times \frac{5}{1{,}000} = 1{,}000 \text{ m}^3$$

Total:

$$WF_{Scope\,3} = 25{,}000 + 40{,}000 + 1{,}000 = 66{,}000 \text{ m}^3$$

Answer: 66,000 m³

Breakdown: - Steel: 37.9% - Plastic: 60.6% - Services: 1.5%

EXAMPLE 10.2

Calculate virtual water content of 1 kg beef with: - Feed: 8 kg grain @ 1,500 L/kg - Drinking water: 50 L - Service water: 100 L - Processing: 200 L

Solution:

Feed water (green + blue):

$$V\,W_{feed} = 8 \times 1{,}500 = 12{,}000 \text{ L}$$

Direct water:

$$V\,W_{direct} = 50 + 100 + 200 = 350 \text{ L}$$

Total virtual water:

$$V\,W_{beef} = 12{,}000 + 350 = 12{,}350 \text{ L/kg} = 12.35 \text{ m}^3\text{/kg}$$

Answer: 12.35 m^3/kg

Note: Feed dominates (97.2% of total), consistent with literature values of 10-15 m^3/kg for beef.

EXAMPLE 10.3

A textile company has: - Scope 1: 50,000 m^3 - Scope 2: 30,000 m^3 - Scope 3 upstream: 200,000 m^3 - Scope 3 downstream: 100,000 m^3

Calculate: (a) Total water footprint, (b) Scope 3 percentage

Solution:

(a) **Total water footprint:**

$$W\,F_{total} = 50{,}000 + 30{,}000 + 200{,}000 + 100{,}000 = 380{,}000 \text{ m}^3$$

Answer (a): 380,000 m^3

(b) **Scope 3 percentage:**

$$\%_{Scope\,3} = \frac{200{,}000 + 100{,}000}{380{,}000} \times 100\,\% = \frac{300{,}000}{380{,}000} \times 100\,\% = 78.9\,\%$$

Answer (b): 78.9%

Interpretation: Scope 3 dominates water footprint (typical for textile industry where cotton cultivation is water-intensive).

EXAMPLE 10.4

Calculate water footprint of a cotton t-shirt (200 g) with: - Cotton production: 10,000 L/kg (mostly green water) - Textile manufacturing: 100 L/kg - Dyeing and finishing: 50 L/kg - Transportation: 5 L/kg - Consumer washing (50 washes): 40 L/wash

Solution:

Production phase:

$$WF_{production} = 0.2 \times (10{,}000 + 100 + 50 + 5) = 0.2 \times 10{,}155 = 2{,}031 \text{ L}$$

Use phase:

$$WF_{use} = 50 \times 40 = 2{,}000 \text{ L}$$

Total life cycle:

$$WF_{LC} = 2{,}031 + 2{,}000 = 4{,}031 \text{ L} = 4.03 \text{ m}^3$$

Answer: 4.03 m^3 per t-shirt

Breakdown: - Cotton growing: 49.6% - Consumer use: 49.6% - Manufacturing: 0.8%

Note: Cotton agriculture and consumer washing are equally important.

EXAMPLE 10.5

A beverage company sells 100 million liters of product. Consumer use requires 2 L water per 1 L product (for cleaning, ice, etc.). Calculate Scope 3 downstream water footprint.

Solution:

Product volume:

$$V_{product} = 100{,}000{,}000 \text{ L} = 100{,}000 \text{ m}^3$$

Consumer water use:

$$W\,F_{consumer} = 100{,}000 \times 2 = 200{,}000 \ \text{m}^3$$

Answer: 200,000 m^3

Note: This is Category 11 (Use of sold products) in Scope 3. For beverages, consumer water use can exceed manufacturing water use.

EXAMPLE 10.6

Using input-output analysis, calculate water footprint of $1 million final demand in automotive sector with: - Direct water intensity: 15 m^3/$1,000 - Leontief inverse element (automotive, automotive): 1.5 - Leontief inverse element (automotive, steel): 0.3 - Steel water intensity: 20 m^3/$1,000

Solution:

Direct water use:

$$W_{direct} = 1{,}000 \times 15 \times 1.5 = 22{,}500 \ \text{m}^3$$

Indirect water use (steel):

$$W_{steel} = 1{,}000 \times 20 \times 0.3 = 6{,}000 \ \text{m}^3$$

Total water footprint:

$$W_{total} = 22{,}500 + 6{,}000 = 28{,}500 \ \text{m}^3$$

Answer: 28,500 m^3

Water intensity: $\frac{28{,}500}{1{,}000} = 28.5$ m^3/$1,000

EXAMPLE 10.7

A region imports 50,000 tonnes wheat (VW = 1.5 m³/kg) and exports 20,000 tonnes manufactured goods (VW = 0.5 m³/kg). Calculate net virtual water import.

Solution:

Virtual water import:

$$VW_{import} = 50{,}000{,}000 \times 1.5 = 75{,}000{,}000\ \mathrm{m}^3 = 75 \times 10^6\ \mathrm{m}^3$$

Virtual water export:

$$VW_{export} = 20{,}000{,}000 \times 0.5 = 10{,}000{,}000\ \mathrm{m}^3 = 10 \times 10^6\ \mathrm{m}^3$$

Net virtual water import:

$$VW_{net} = 75 - 10 = 65 \times 10^6\ \mathrm{m}^3$$

Answer: 65 million m³ net import

Interpretation: The region effectively imports 65 million m³ of water through trade, reducing pressure on local water resources.

EXAMPLE 10.8

Calculate Scope 3 water footprint for business travel: 500 employees × 5 nights/year × 300 L/night hotel water use.

Solution:

$$WF_{travel} = 500 \times 5 \times 300 = 750{,}000\ \mathrm{L} = 750\ \mathrm{m}^3$$

Answer: 750 m³

Note: This is Category 6 (Business travel) in Scope 3. Relatively small compared to other categories but should be included for completeness.

EXAMPLE 10.9

A food company has Scope 1+2 = 100,000 m^3. Agricultural supply chain (Scope 3) has water intensity of 2,000 m^3/tonne and company purchases 10,000 tonnes/year. Calculate total footprint and Scope 3 percentage.

Solution:

Scope 3:

$$WF_{Scope\,3} = 10{,}000 \times 2{,}000 = 20{,}000{,}000 \ \mathrm{m}^3$$

Total:

$$WF_{total} = 100{,}000 + 20{,}000{,}000 = 20{,}100{,}000 \ \mathrm{m}^3$$

Scope 3 percentage:

$$\%_{Scope\,3} = \frac{20{,}000{,}000}{20{,}100{,}000} \times 100\,\% = 99.5\,\%$$

Answer: Total = 20.1 million m^3, Scope 3 = 99.5%

Interpretation: For food companies, agricultural supply chain water use typically dominates (>95% of total).

EXAMPLE 10.10

Calculate end-of-life water footprint for 1,000 tonnes of product with: - 60% recycled @ 5 m^3/tonne water use - 30% landfilled @ 1 m^3/tonne leachate treatment - 10% incinerated @ 2 m^3/tonne flue gas treatment

Solution:

Recycling:

$$W\,F_{recycle}=1{,}000\times0.6\times5=3{,}000\ \text{m}^3$$

Landfill:

$$W\,F_{landfill}=1{,}000\times0.3\times1=300\ \text{m}^3$$

Incineration:

$$W\,F_{incineration}=1{,}000\times0.1\times2=200\ \text{m}^3$$

Total EOL water footprint:

$$W\,F_{EOL}=3{,}000+300+200=3{,}500\ \text{m}^3$$

Answer: 3,500 m^3

Per tonne: 3.5 m^3/tonne

SUPPLEMENTARY PROBLEMS

10.1 Company purchases 5,000 tonnes material @ WI = 30 m^3/tonne. Find Scope 3 WF. **Ans.** 150,000 m^3

10.2 Virtual water: 1 kg rice = 2.5 m^3. Region imports 100,000 tonnes. Find VW import. **Ans.** 250,000 m^3

10.3 Life cycle WF: Production = 50,000 m^3, Use = 80,000 m^3, EOL = 5,000 m^3. Find total. **Ans.** 135,000 m^3

10.4 Scope 1 = 40,000, Scope 2 = 20,000, Scope 3 = 240,000 m^3. Find Scope 3 %. **Ans.** 80%

10.5 Hotel: 200 L/night. Business travel: 1,000 room-nights. Find WF. **Ans.** 200 m^3

10.6 Beef: 15 m^3/kg. Company purchases 500 kg/year. Find Scope 3 WF. **Ans.** 7,500 m^3

10.7 T-shirt: Cotton = 2,000 L, Manufacturing = 150 L, Use = 2,000 L. Find total. **Ans.** 4,150 L

10.8 Import 20,000 t @ VW = 2 m^3/kg, export 10,000 t @ VW = 0.8 m^3/kg. Find net VW. **Ans.** 32,000,000 m^3

10.9 Spend-based: \$500,000 @ WI = 8 m^3/\$1,000. Find WF. **Ans.** 4,000 m^3

10.10 Recycling: 2,000 t @ 4 m^3/t. Landfill: 500 t @ 1 m^3/t. Find total EOL WF. **Ans.** 8,500 m^3

Chapter 11: WATER PRODUCTIVITY AND EFFICIENCY

11.1 WATER PRODUCTIVITY METRICS

Physical Water Productivity

Definition: Output per unit of water consumed.

$$WP_{physical} = \frac{Y}{W_{cons}}$$

where: - Y = Physical output (kg, tonnes, units) - W_{cons} = Consumptive water use (m³)

Units: kg/m³, tonnes/m³

Economic Water Productivity

Definition: Economic value per unit of water consumed.

$$WP_{economic} = \frac{V}{W_{cons}}$$

where: - V = Economic value (currency) - W_{cons} = Consumptive water use (m³)

Units: \$/m³, €/m³

"Crop Per Drop"

For agriculture:

$$WP_{crop} = \frac{Y}{ET_c}$$

where: - Y = Crop yield (kg/ha) - ET_c = Crop evapotranspiration (m³/ha)

Typical Values: - Wheat: 0.8-1.2 kg/m³ - Rice: 0.6-1.0 kg/m³ - Maize: 1.0-1.8 kg/m³ - Cotton: 0.3-0.5 kg/m³

11.2 WATER USE EFFICIENCY

Irrigation Efficiency

Application Efficiency:

$$E_a = \frac{W_{stored}}{W_{applied}} \times 100\%$$

where: - W_{stored} = Water stored in root zone (m³) - $W_{applied}$ = Water applied by irrigation (m³)

Typical Values: - Surface irrigation: 50-70% - Sprinkler: 70-85% - Drip irrigation: 85-95%

Conveyance Efficiency

$$E_c = \frac{W_{delivered}}{W_{diverted}} \times 100\%$$

Typical Values: - Unlined canals: 60-70% - Lined canals: 80-90% - Pipes: 95-99%

Overall Irrigation Efficiency

$$E_{overall} = E_c \times E_a$$

11.3 WATER INTENSITY

Water Intensity (Inverse of Productivity)

$$WI = \frac{W}{Y} = \frac{1}{\wp}$$

Units: m³/kg, m³/unit, m³/$

Water Intensity Benchmarking

Percentile Analysis:

- P10 (best 10%): Leading performers
- P25: Good performance
- P50 (median): Average performance

- P75: Below average
- P90 (worst 10%): Poor performance

Improvement Potential

$$IP = \frac{WI_{current} - WI_{best}}{WI_{current}} \times 100\%$$

11.4 RETURN FLOWS AND BASIN EFFICIENCY

Consumptive vs. Non-Consumptive Use

Consumptive Use:

$$W_{cons} = W_{evap} + W_{transp} + W_{incorp}$$

Non-Consumptive Use:

$$W_{non-cons} = W_{withdrawal} - W_{cons} = W_{return}$$

Basin-Level Efficiency

At basin scale, return flows may be reused:

$$E_{basin} = \frac{W_{beneficial}}{W_{diverted}}$$

where $W_{beneficial}$ includes all beneficial uses (crop ET, municipal use, industrial use, environmental flows).

Theorem 11.1 (Field vs. Basin Efficiency Paradox)

Statement:

Improving field-level irrigation efficiency does not necessarily improve basin-level water availability if return flows are reused downstream.

Proof:

Field Level:

$$W_{cons,field} = W_{applied} \times E_a$$

Improving E_a reduces $W_{applied}$ for same $W_{cons,field}$.

Basin Level:

Return flow: $W_{return} = W_{applied} \times (1 - E_a)$

If downstream users depend on W_{return}, reducing $W_{applied}$ by improving E_a reduces water available to downstream users.

Net basin consumption remains unchanged:

$$W_{cons,basin} = W_{cons,field} + W_{cons,downstream}$$

∎

Practical Implication:

"Saving water" at field level may not save water at basin level. Must consider return flow reuse.

11.5 BENCHMARKING AND TARGET SETTING

Data Envelopment Analysis (DEA)

DEA identifies efficient frontier of water use:

$$\max \theta$$

subject to:

$$\sum_{j=1}^{n} \lambda_j y_j \geq y_i$$

$$\sum_{j=1}^{n} \lambda_j w_j \leq \theta w_i$$

$$\lambda_j \geq 0$$

where: - θ = Efficiency score (0 to 1) - y_j = Output of facility j - w_j = Water use of facility j

Target Setting

Absolute Target:

$$W_{target} = W_{baseline} \times (1 - r)$$

where r = reduction rate (e.g., 0.20 for 20% reduction)

Intensity Target:

$$WI_{target} = WI_{baseline} \times (1 - r)$$

Allows for production growth while improving efficiency.

SOLVED PROBLEMS

EXAMPLE 11.1

A farm produces 8,000 kg wheat using 10,000 m³ irrigation water. Calculate: (a) Physical water productivity, (b) Water intensity.

Solution:

(a) **Physical water productivity:**

$$\wp = \frac{8{,}000}{10{,}000} = 0.8 \text{ kg/m}^3$$

Answer (a): 0.8 kg/m³

(b) **Water intensity:**

$$WI = \frac{10{,}000}{8{,}000} = 1.25 \text{ m}^3/\text{kg}$$

Answer (b): 1.25 m³/kg

Note: $WI = 1/\wp$ ✓

EXAMPLE 11.2

A factory produces \$5 million revenue using 50,000 m³ water. Calculate economic water productivity.

Solution:

$$WP_{economic} = \frac{5{,}000{,}000}{50{,}000} = 100\ \$/m^3$$

Answer: \$100/m³

Interpretation: Each cubic meter of water generates \$100 in revenue.

EXAMPLE 11.3

Irrigation system: 1,000 m³ diverted, 900 m³ delivered, 630 m³ stored in root zone. Calculate: (a) Conveyance efficiency, (b) Application efficiency, (c) Overall efficiency.

Solution:

(a) **Conveyance efficiency:**

$$E_c = \frac{900}{1{,}000} \times 100\% = 90\%$$

Answer (a): 90%

(b) **Application efficiency:**

$$E_a = \frac{630}{900} \times 100\% = 70\%$$

Answer (b): 70%

(c) **Overall efficiency:**

$$E_{overall} = E_c \times E_a = 0.90 \times 0.70 = 0.63 = 63\%$$

Answer (c): 63%

Interpretation: Only 63% of diverted water is stored in root zone; 37% is lost to seepage and runoff.

EXAMPLE 11.4

Current water intensity: 25 m^3/unit. Best-in-class: 15 m^3/unit. Calculate improvement potential.

Solution:

$$IP = \frac{25-15}{25} \times 100\% = \frac{10}{25} \times 100\% = 40\%$$

Answer: 40% improvement potential

Interpretation: By adopting best practices, water use could be reduced by 40%.

EXAMPLE 11.5

Baseline water use: 100,000 m^3. Set 25% reduction target over 5 years. What is annual target?

Solution:

Final target:

$$W_{target} = 100{,}000 \times (1-0.25) = 75{,}000 \text{ m}^3$$

Annual reduction:

$$\Delta W_{annual} = \frac{100{,}000 - 75{,}000}{5} = 5{,}000 \text{ m}^3/\text{year}$$

Answer: Reduce by 5,000 m^3/year to reach 75,000 m^3 by year 5

Annual targets: - Year 1: 95,000 m^3 - Year 2: 90,000 m^3 - Year 3: 85,000 m^3 - Year 4: 80,000 m^3 - Year 5: 75,000 m^3

SUPPLEMENTARY PROBLEMS

11.1 Yield = 12,000 kg, Water = 15,000 m³. Find WP. **Ans.** 0.8 kg/m³

11.2 Revenue = $2M, Water = 40,000 m³. Find economic WP. **Ans.** $50/m³

11.3 Diverted = 500 m³, Delivered = 450 m³, Stored = 360 m³. Find overall efficiency. **Ans.** 72%

11.4 Current WI = 30 m³/unit, Best = 18 m³/unit. Find improvement potential. **Ans.** 40%

11.5 Baseline = 80,000 m³. 30% reduction target. Find target. **Ans.** 56,000 m³

Chapter 12: WATER RISK ASSESSMENT

12.1 WATER STRESS INDICATORS

Falkenmark Indicator

Water Stress Threshold:

$$WS = \frac{W_{available}}{P}$$

where: - $W_{available}$ = Annual renewable water resources (m³) - P = Population

Classification: - $WS > 1{,}700$ m³/person/year: No stress - $1{,}000 < WS < 1{,}700$: Stress - $500 < WS < 1{,}000$: Scarcity - $WS < 500$: Absolute scarcity

Water Withdrawal-to-Availability Ratio

$$WSI = \frac{W_{withdrawal}}{W_{available}} \times 100\%$$

Classification: - $WSI < 10\%$: Low stress - $10\% < WSI < 20\%$: Low-medium stress - $20\% < WSI < 40\%$: Medium-high stress - $WSI > 40\%$: High stress

Baseline Water Stress (WRI Aqueduct)

$$BWS = \frac{W_{withdrawal}}{W_{available} - EFR}$$

where EFR = Environmental flow requirement

12.2 WATER RISK CATEGORIES

Physical Risk

- **Quantity:** Scarcity, drought, variability
- **Quality:** Pollution, salinity, eutrophication
- **Ecosystems:** Degradation of water-dependent ecosystems

Regulatory Risk

- **Pricing:** Water tariff increases
- **Allocation:** Reduced water rights
- **Quality standards:** Stricter discharge limits
- **Reporting:** Mandatory disclosure requirements

Reputational Risk

- **Community opposition** to water use
- **NGO campaigns** targeting water practices
- **Media coverage** of water impacts
- **Consumer boycotts**

12.3 WATER RISK ASSESSMENT FRAMEWORK

Risk Matrix

$$Risk = Likelihood \times Impact$$

Likelihood Scale (1-5): 1. Rare (< 10% probability) 2. Unlikely (10-30%) 3. Possible (30-50%) 4. Likely (50-70%) 5. Almost certain (> 70%)

Impact Scale (1-5): 1. Negligible (< $10k) 2. Minor ($10k-$100k) 3. Moderate ($100k-$1M) 4. Major ($1M-$10M) 5. Severe (> $10M)

Risk Score: 1-25 (Likelihood × Impact)

Risk Categories: - 1-5: Low risk (green) - 6-12: Medium risk (yellow) - 13-25: High risk (red)

12.4 ENVIRONMENTAL FLOW REQUIREMENTS

Tennant Method (Montana Method)

Minimum Flow:

$$Q_{min} = 0.10 \times Q_{mean}$$

Fair Flow:

$$Q_{fair} = 0.30 \times Q_{mean}$$

Good Flow:

$$Q_{good} = 0.50 \times Q_{mean}$$

where Q_{mean} = Mean annual flow

Presumptive Environmental Flow Standard

$$EFR = \begin{cases} 0.60 \times Q_{mean} & \text{high biodiversity} \\ 0.45 \times Q_{mean} & \text{moderate biodiversity} \\ 0.30 \times Q_{mean} & \text{low biodiversity} \end{cases}$$

12.5 WATER-RELATED FINANCIAL RISK

Value at Risk (VaR)

Definition: Maximum loss at given confidence level.

For water-related disruption with probability p and impact I:

$$VaR_{95\%} = I \times \Phi^{-1}(0.95)$$

where Φ^{-1} is the inverse cumulative distribution function.

Expected Loss

$$E[Loss] = \sum_i p_i \times I_i$$

where: - p_i = Probability of scenario i - I_i = Financial impact of scenario i

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SOLVED PROBLEMS

EXAMPLE 12.1

A basin has 50 km³/year renewable water, population 40 million. Calculate Falkenmark indicator and classify stress level.

Solution:

$$WS = \frac{50 \times 10^9}{40 \times 10^6} = 1{,}250 \text{ m}^3\text{/person/year}$$

Answer: 1,250 m³/person/year = **Water Stress** (between 1,000 and 1,700)

Interpretation: The basin is under water stress and should prioritize water conservation.

EXAMPLE 12.2

Annual withdrawal: 20 km³. Available water: 60 km³. Calculate WSI and classify.

Solution:

$$WSI = \frac{20}{60} \times 100\% = 33.3\%$$

Answer: 33.3% = **Medium-High Stress** (between 20% and 40%)

Interpretation: Significant water stress; demand management needed.

EXAMPLE 12.3

Water risk assessment: Drought likelihood = 4 (likely), Financial impact = $5M (major, score 4). Calculate risk score and category.

Solution:

$$Risk = 4 \times 4 = 16$$

Answer: Risk score = 16 = **High Risk** (13-25 range, red category)

Recommendation: Develop drought contingency plan and diversify water sources.

EXAMPLE 12.4

Mean annual flow: 100 m^3/s. Calculate environmental flow requirement using Tennant method for "good" conditions.

Solution:

$$Q_{good} = 0.50 \times 100 = 50 \ m^3/s$$

Answer: 50 m^3/s

Interpretation: At least 50 m^3/s must be maintained in the river for good ecological conditions.

EXAMPLE 12.5

Three water risk scenarios: - Drought: 20% probability, \$2M impact - Regulation: 30% probability, \$500k impact - Pollution: 10% probability, \$1M impact

Calculate expected loss.

Solution:

$$E[Loss] = 0.20 \times 2{,}000{,}000 + 0.30 \times 500{,}000 + 0.10 \times 1{,}000{,}000$$

$$E[Loss] = 400{,}000 + 150{,}000 + 100{,}000 = 650{,}000$$

Answer: \$650,000 expected annual loss

Interpretation: Budget at least \$650k/year for water risk management.

SUPPLEMENTARY PROBLEMS

12.1 Water available = 80 km^3, Population = 50M. Find Falkenmark indicator. **Ans.** 1,600 m^3/person/year (stress)

12.2 Withdrawal = 15 km^3, Available = 40 km^3. Find WSI. **Ans.** 37.5% (medium-high stress)

12.3 Likelihood = 3, Impact = 5. Find risk score. **Ans.** 15 (high risk)

12.4 Mean flow = 200 m^3/s. Find minimum environmental flow (Tennant). **Ans.** 20 m^3/s

12.5 Two scenarios: (1) 40% prob, \$1M impact; (2) 10% prob, \$5M impact. Find expected loss. **Ans.** \$900,000

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Data and Tools

- **Water Footprint Network:** https://www.waterfootprint.org/
- **Aqueduct Water Risk Atlas (WRI):** https://www.wri.org/aqueduct
- **FAO AQUASTAT:** http://www.fao.org/aquastat/
- **CDP Water Security:** https://www.cdp.net/en/water
- **CEO Water Mandate:** https://ceowatermandate.org/
- **Alliance for Water Stewardship:** https://a4ws.org/

Software

- **SWAT (Soil and Water Assessment Tool):** https://swat.tamu.edu/
- **WEAP (Water Evaluation and Planning):** https://www.weap21.org/
- **OpenLCA:** https://www.openlca.org/

Standards and Protocols

- **GHG Protocol:** https://ghgprotocol.org/
- **ISO 14046 Water Footprint:** https://www.iso.org/standard/43263.html
- **SEEA-Water:** https://seea.un.org/content/seea-water

GLOSSARY OF ABBREVIATIONS

- **AWS** - Alliance for Water Stewardship
- **BOD** - Biochemical Oxygen Demand
- **CDP** - Carbon Disclosure Project (now CDP)
- **CN** - Curve Number (SCS method)
- **DO** - Dissolved Oxygen
- **ET** - Evapotranspiration
- **ET_0** - Reference Evapotranspiration
- **ETc** - Crop Evapotranspiration
- **FAO** - Food and Agriculture Organization
- **GHG** - Greenhouse Gas
- **HRU** - Hydrological Response Unit
- **IPCC** - Intergovernmental Panel on Climate Change
- **ISO** - International Organization for Standardization
- **Kc** - Crop Coefficient
- **LCA** - Life Cycle Assessment
- **NRCS** - Natural Resources Conservation Service (USDA)
- **OECD** - Organisation for Economic Co-operation and Development
- **SCS** - Soil Conservation Service (now NRCS)
- **SEEA** - System of Environmental-Economic Accounting
- **SWAT** - Soil and Water Assessment Tool
- **UN** - United Nations
- **USDA** - United States Department of Agriculture
- **WEAP** - Water Evaluation and Planning
- **WF** - Water Footprint
- **WFN** - Water Footprint Network
- **WQI** - Water Quality Index

- **WRI** - World Resources Institute
- **WSI** - Water Stress Index

INDEX

(Note: A comprehensive index would be generated during final typesetting. Key terms to include:)